MthT 430 Summary 2007[1]

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This document will be updated as the semester progresses.

MthT 430, *Mathematical Analysis For Teachers I*, is offered at the University of Illinois at Chicago for undergraduates and graduate students. The text is Michael Spivak's **Calculus**, which is also used for the mathematics senior analysis sequence.

The course was first developed in Fall 2001 and 2002 by Jeff Lewis with the assistance of Kyungsoon Jeon.

This is a brief outline of the theory developed in Chapters 1–10 of Spivak's **Calculus**, Third Edition.

I indicate some affiliated activities.

Theory

Ordered Field

- Addition + (P1 – P4) (Commutative Group)
- Multiplication · (P5 – P8)
- Distributive – Glue (P9)

Activity: Fill in addition and multiplication tables for examples which look like $\mathbf{Z_2}$.

$$a \cdot 0 = 0,$$

$$-(a \cdot b) = (-a) \cdot b.$$

- Order (P10 – P12)
  1 is positive
  Meaning of $>, \geq, <, \leq$.

Activity: Finite fields can not be ordered. $1 > 0$. Complex numbers can not be ordered. Rational and irrational numbers.

Inequality Lemma. $a \leq b$ if and only if: For every $\epsilon > 0$, $a < b + \epsilon$.
Definition: A set A is bounded [above, below].

- Absolute Value and Triangle Inequality

[1] Jeff E. Lewis

Mathematical Induction and Recursion

- Statement of Principle of Mathematical Induction (PMI) and Well Ordering Principle (WOP)

- Examples

  Define the binary (decimal) expansion of a number by recursion.

  The sequence $\{2^n\}$ is not bounded above.

Activity: Tower of Hanoi. Definition by recursion.

Functions

- Definition as a set of ordered pairs

- Domain of a function

- Extension of a function

- Conventions; Implied domain for formulas.

- Sums, products, compositions and domains of such.

- Sequences: A *sequence* $\{a_n\}$ is a function a whose domain

$$\begin{cases} \text{is a subset of the integers,} & (1) \\ \text{includes all natural numbers, } n\text{, sufficiently large.} & (2) \end{cases}$$

Write $a(n) = a_n$.

Limits

- Define (ϵ–δ): $\lim_{x \to a} f(x) = L$.

- Implications about domain.

- Variations: Left limit, right limit, $\lim_{x \to \infty} f(x)$.

- Infinite limits: $\lim_{x \to *} f(x) = \pm\infty$

- Limit Theorems: sum, product, and quotient (with caveat).

Activity: Correct (equivalent) definitions. Incorrect definitions. Notation exchange. Monotone functions have one sided limits.

Continuous Functions

- Continuous at a point a: $\lim_{x\to a} f(x) = f(a)$. Implications about domain$f$. Statement of continuity [at a point a] in terms of ϵ–δ.

- Continuous from the left [right] at a point a: $\lim_{x\to a^-[a^+]} f(x) = f(a)$. Implications about domain$f$.

- Continuous on a particular interval $[a,b]$, (a,b), $[a,b)$, $(a,b]$.

  Language: The function f is continuous on $[0,1)$ means f is continuous from the right at 0 and for all a, $0 < a < 1$, f is continuous at a.

  Activity: Continuity in the context of rationals.

- Examples continuous at every point in their whole domain: x, x^2, $\dfrac{1}{x}$. The square root function, $\sqrt{x}$, is continuous on $[0,\infty)$.

- Theorems: Sum and product of continuous functions are continuous (at a point and/or on an interval). Quotient with caveat.

- **Composition Theorem**. If g is [left,right, both sides] continuous at a, and f is continuous at $g(a)$, then the composition $f\circ g$ is [left,right,both sides] continuous at a.

- **Composition Theorem on Intervals**. If g is continuous on an interval I, and f is continuous on an interval containing $g(I)$, then the composition $f\circ g$ is continuous on I.

Completeness

- Definition of *bounded above, bounded below, bounded, least upper bound* (sup)*, greatest lower bound* (inf)*, greatest element, least element*.

- **Definition**. A number x is a **least upper bound** for a nonempty set A if

$$\begin{cases} x \text{ is an upper bound for } A, & (1) \\ \text{if } y \text{ is an upper bound for } A, \text{ then } x \le y. & (2) \end{cases}$$

  Such a number x is also called the **supremum** for A and sometimes denoted by $\sup A$ or lub$A$.

  Activity: Binary expansion. Monotone sequences and functions.

- **Definition** (Alternate). A number x is a **least upper bound** for a nonempty set A if

$$\begin{cases} x \text{ is an upper bound for } A, & (1) \\ \text{For every } \epsilon > 0, \text{ there is an } x_\epsilon \in A \text{ such that } x-\epsilon < x_\epsilon \le x. & (2') \end{cases}$$

- Three Hard Theorems

  ◇ CFCIVP: Continuous functions on closed intervals have the intermediate value property.

  ◇ CFCIB: Continuous functions on closed intervals are bounded on the interval.

  ◇ CFCIMAX: Continuous functions on closed intervals assume MAXimum and MINinimum values on the interval.

  Activity: Using the binary expansion.

- (P13–BISHL) Bounded increasing sequences have limits. If $\{a_n\}$ is a bounded increasing [nondecreasing] sequence, $\lim_{n\to\infty} a_n = \sup_n a_n$. Equivalent to (P13).

- (P13–BIN) Every binary expansion represents a real number. Equivalent to (P13).

- Monotone functions have right and left limits: If for some $\delta > 0$, f is defined and nondecreasing on $(a-\delta, a+\delta)$, then

$$\begin{aligned}\lim_{x\to a^-} f(x) &= \sup_{(a-\delta,a)} f(x)\\ &\le f(a)\\ &\le \inf_{(a,a+\delta)} f(x)\\ &= \lim_{x\to a^+} f(x).\end{aligned}$$

  Activity: What happens if all numbers were rational?

Derivatives

- Definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

- f' is a function with domain ...

- Differentiability and continuity

Activity: Derivatives of common functions

Differentiation

- Linear approximation

- Sums, products, and composition

Activity: Practice