MthT 430 Notes Chapter 10 Differentiation

The purpose of learning differentiation technique is to develop formulas for almost any function defined by a formula.

From the **Chapter 5, Limit Theorems**, we easily obtain for *sums (differences)* of differentiable functions:

Theorem 3. *If f and g are differentiable at a, then $f \pm g$ is differentiable at a, and*

$$(f \pm g)'(a) = f'(a) \pm g'(a).$$

More briefly,

Theorem 3. *If f and g are differentiable [on a set I], then $f \pm g$ is differentiable [on I], and*

$$(f \pm g)' = f' \pm g'.$$

Product Rule for Differentiation

Theorem 4. *If f and g are differentiable at a, then*

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Proof. *p. 167.*

The idea behind the proof: Let $\Delta_h f = f(a+h) - f(a)$, etc., so that

$$f'(a) = \lim_{h \to 0} \frac{\Delta_h f}{h},$$

etc.,

$$\begin{aligned}
(f \cdot g)(a+h) &= (f(a) + \Delta_h f) \cdot (g(a) + \Delta_h g) \\
&= f(a) \cdot g(a) + \Delta_h f \cdot g(a) + f(a) \cdot \Delta_h g + \Delta_h f \cdot \Delta_h g, \\
\Delta_h (f \cdot g) &= (f \cdot g)(a+h) - f(a) \cdot g(a) \\
&= \Delta_h f \cdot g(a) + f(a) \cdot \Delta_h g + \Delta_h f \cdot \Delta_h g, \\
\lim_{h \to 0} \frac{\Delta_h (f \cdot g)}{h} &= \lim_{h \to 0} \frac{\Delta_h f}{h} \cdot g(a) + f(a) \cdot \lim_{h \to 0} \frac{\Delta_h g}{h} + \lim_{h \to 0} \frac{\Delta_h f}{h} \cdot \lim_{h \to 0} \Delta_h g \\
&= f'(a) \cdot g(a) + f(a) \cdot g'(a).
\end{aligned}$$

The derivative of the n^{th} *power function* is given by:

Theorem 6. *If $f(x) = x^n$ for some natural number n, then*

$$f'(a) = na^{n-1} \tag{♠}$$

for all a.

Derivative of a Quotient

Theorem 7. *If g is differentiable at a, and $g(a) \neq 0$, then $1/g$ is differentiable at a, and*

$$(1/g)'(a) = -\frac{g'(a)}{(g(a))^2}.$$

Proof. p. 167. An alternate proof is to observe that $1/g$ is the composition of the *oneover function* with g and use the *chain rule*.

Theorem 8 (Quotient Rule). *If f and g is differentiable at a, and $g(a) \neq 0$, then f/g is differentiable at a, and*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - f(a) \cdot g'(a)}{|g(a)|^2}.$$

An alternate form:

$$\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2}.$$

From an old calculus book (Robert Bonic, et al., **Freshman Calculus**):

$$\left(\frac{u}{v}\right)' : \frac{}{v^2} \qquad \text{(write vinculum over } v^2\text{)}$$

$$\to \frac{v}{v^2} \qquad \text{(write } v \text{ again before you forget)}$$

$$\to \frac{vu' - v'u}{v^2} \qquad \text{(fill in the rest)}$$

Some prefer the forms:

$$\begin{aligned}\left(\frac{u}{v}\right)' &= \left\{u \cdot \left(\frac{1}{v}\right)\right\}' \\ &= \frac{u'}{v} - u \cdot \left(\frac{v'}{v^2}\right),\end{aligned}$$

$$\left(uv^{-1}\right)' = u'v^{-1} + (-1)\,uv'v^{-2}. \tag{♣}$$

The form (♣) is particularly convenient for repeated derivatives.

Chain Rule

Theorem 9 (Chain Rule). *If g is differentiable at a, and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Other forms of the Chain Rule

Leibniz notation:

$$y = y(u),$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Leibniz (mixed) notation:

$$\frac{df(u(x))}{dx} = f'(u(x)) \cdot \frac{du}{dx}.$$

Function notation:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

One layer at a time:

$$\begin{aligned}
\frac{df(\heartsuit(\ldots))}{dx} &: f'(\qquad) && \text{(deriv of outside function } f\text{)}\\
&\to f'(\heartsuit(\ldots)) && \text{(evaluate at inside function)}\\
&\to f'(\heartsuit(\ldots))\cdot && \text{(}\textbf{TIMES}\text{)}\\
&\to f'(\heartsuit(\ldots)) \cdot \left(\frac{d(\heartsuit(\ldots))}{dx}\right) && \text{(deriv of inside)}\\
& && \text{(do it again if necessary)}
\end{aligned}$$