MthT 430 Notes Chapter 1a More on Basic Properties of Numbers

Properties (P1) – (P9) give the following Theorems:

Theorem. *Assuming P1 – P9, for all a,* $a \cdot 0 = 0$.

The proof is given on page 7 of Spivak's book.

Proof.

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) \\ &= a \cdot 0 + a \cdot 0. && \text{(P9)} \end{aligned}$$

Now subtract $a \cdot 0$ from both sides of the equation

$$a \cdot 0 = a \cdot 0 + a \cdot 0.$$

Theorem. *If*

$$a \cdot b = 0,$$

then either

$$a = 0 \text{ or } b = 0.$$

Proof. If $a \neq 0$ and $a \cdot b = 0$, then

$$\begin{aligned} a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot 0, \\ \left(a^{-1} \cdot a\right) \cdot b &= 0, \\ 1 \cdot b &= 0, \\ b &= 0. \end{aligned}$$

Similarly, if $b \neq 0$, then $a = 0$.

Theorem.

$$\begin{aligned} -(a \cdot b) &= (-a) \cdot b \\ &= a \cdot (-b)\,. \end{aligned}$$

Proof. We will show that

$$(a \cdot b) + (-a) \cdot b = 0.$$

$$\begin{aligned} (a \cdot b) + (-a) \cdot b &= (a + (-a)) \cdot b && \text{(by P9)} \\ &= 0 \cdot b && \text{(by P3)} \\ &= 0 && \text{(by Theorem).} \end{aligned}$$

The characterization of $-(a \cdot b)$ as a $\cdot (-b)$ is shown by repeating the proof or interchanging the roles of a and b and using (P8).

Inequalities

Inequalities are defined in terms of of the positive numbers P. We say that $a > 0$ or $0 < a$ if a is in P. We that $a < b$ $[b > a]$ if $0 < b - a$ or $b - a$ is in P.

Note that

- If $a < b$, then for all c, $a + c < b + c$.
- If $a < b$, and $0 < c$, then $ac < bc$.
- If $a < b$, and $c < 0$, then $ac > bc$.

We shall also say that a is *nonnegative* $[a \geq 0]$ if and only if $a = 0$ *or* $a > 0$.

- For all a, a^2 is nonnegative.
- If a and b are nonnegative, then $a \leq b$ iff $a^2 \leq b^2$.

Absolute Value

Definition. The absolute value of a number a is defined as

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Properties of absolute value often involve a *proof by cases*.

Theorem. *For all* a,

$$-|a| \leq a \leq |a|.$$

Also

$$-|a| \leq -a \leq |a|.$$

Proof. If $a \geq 0$, then $|a| = a$, and

$$-|a| \leq 0 \leq a = |a|.$$

If $a \leq 0$, then $-|a| = a$, and

$$-|a| = a \leq 0 \leq |a|.$$

The second statement is shown in a similar manner, or by multiplying each expression in the first statement by -1.

Theorem. *(The triangle inequality) For all numbers a, b,*

$$|a+b| \leq |a| + |b| .$$

Proof. We have

$$\begin{aligned} -|a| \leq a \leq |a| , \\ -|b| \leq b \leq |b| , \end{aligned}$$

so that

$$-(|a| + |b|) \leq a + b \leq |a| + |b| .$$

Now break into the two cases $a + b \geq 0$ and $a + b \leq 0$.

Another Proof. Since $|a+b|$ and $|a| + |b|$ are both nonnegative, compare $|a+b|^2$ and $(|a| + |b|)^2$.

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2, \\ (|a| + |b|)^2 &= a^2 + 2|a||b| + b^2. \end{aligned}$$

We know(?) that $|ab| = |a||b|$ so that

$$\begin{aligned} 2ab \leq 2|a||b| , \\ |a+b|^2 \leq (|a| + |b|)^2 . \end{aligned}$$

Fractions

For this section assume (P1) – (P9).

If $b \neq 0$, we define

$$\begin{aligned} \frac{1}{b} &\equiv b^{-1}, \\ \frac{a}{b} &\equiv a \cdot b^{-1} \\ &= a \cdot \frac{1}{b}. \end{aligned}$$

Of course we hope that for $b \neq 0$, $d \neq 0$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Theorem. *For $b \neq 0$, $d \neq 0$,*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Proof. We show that

$$
\begin{aligned}
(ad+bc)\cdot(bd)^{-1} &= ab^{-1}+cd^{-1}. \\
(ad+bc)\cdot(bd)^{-1} &= (ad+bc)\cdot d^{-1}b^{-1} && \text{why?} \\
&= a\cdot d\cdot d^{-1}\cdot b^{-1}+b\cdot c\cdot d^{-1}\cdot b^{-1} && \text{by P9} \\
&= a\cdot 1\cdot b^{-1}+1\cdot c\cdot d^{-1} && \text{by P8 and P7} \\
&= ab^{-1}+cd^{-1}. && \text{by P9}
\end{aligned}
$$