MthT 430 Notes Chapter 2b Induction etc.

Proofs by Mathematical Induction

PMI is also stated as

Suppose $P(n)$ is a statement for each natural number n. If

$$\begin{cases} P(1) \text{ is true,} \\ \text{Whenever } P(k) \text{ is true, } P(k+1) \text{ is true.} \end{cases}$$

then

$$P(n) \text{ is true for all } n \in \mathbf{N}.$$

A proof using PMI requires:

- Carefully stating the *proposition* or *statement* $P(k)$, which may be a sentence (paragraph) or equation, inequality, ..., which depends on k.
- Proving that $P(1)$ is true – usually a simple verification.
- Assume $P(k)$[1] – write down the statement or formula $P(k)$:
  - Assume ... (statement or formula which depends on k).
  - Apply valid operations (theorems, add to both sides of an equation, etc.) ...

Remember that $P(1)$ is valid too. Try to arrive at $P(k+1)$ as a statement or equation.

Examples

1. Prove the formula

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- $P(n)$ is the sentence

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

What is the verb of the sentence?

[1] As a variant, assume that $P(1), \ldots, P(k)$, are true and use PCI.

- $P(1)$ is true means to verify the statement

$$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}.$$

- $P(n)$ *is true* implies $P(n+1)$ *is true* means to assume

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

and to show that

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)\left((n+1)+1\right)(2\left(n+1\right)+1)}{6}$$

The *valid operations* are adding $(n+1)^2$ to both sides of the equation $P(n)$ and various algebraic manipulations.

2. Tower of Hanoi

- $P(n)$ is the statement:

  - n rings can be moved from one spindle to another with $2^n - 1$ moves, and no fewer.

Other manageable forms of $P(n)$ are

  - The minimal number of moves to move n rings from one spindle to another is $2^n - 1$.

  - It takes $2^n - 1$ moves to move n rings from one spindle to another.

- $P(1)$ is easy.

- $P(n)$ *is true* implies $P(n+1)$ *is true* means to assume *n rings can be moved from one spindle to another with $2^n - 1$ moves, and no fewer*, and to show that *$n+1$ rings can be moved from one spindle to another with $2^{n+1} - 1$ moves, and no fewer*.

In this case, the *valid operations* are a precise sequence of moves to move the $n+1$ rings – the top n rings are moved using $P(n)$, the bottom ring is moved and then the top rings are moved a second time using $P(n)$.

More on the Tower of Hanoi

Let $M(k)$ be the minimal number of moves.

- $P(k)$: $M(k) = 2^k - 1$.
- Verify $P(1)$: $M(1) = 2^1 - 1 = 1$.

Move the single ring from 1 to 3.

- Assume $P(k)$: $M(k) = 2^k - 1$.

Now let there be $k+1$ rings be stacked on pole 1. The bottom ring can be moved only to an empty pole. So the bottom ring can be moved from Pole 1 to Pole 3 iff Pole 3 is empty! Therefore moving the bottom ring from Pole 1 [2] to Pole 3 requires that the top k rings be stacked on Pole 2 [1] – this operation requires $M(k) = 2^k - 1$ moves. So the minimal number of moves for $k + 1$ rings is determined by the following *valid operations*:

1. Move the top k rings from Pole 1 to Pole 2 using $M(k)$ moves,
2. Move the bottom ring from Pole 1 to Pole 3 in 1 move,
3. Move the k rings from Pole 2 to Pole 3 using $M(k)$ moves.

The strategy is minimal since moving the bottom ring from Pole 1 to Pole 3 requires at least $M(k) + 1$ moves.

Then, using $P(k)$, $M(k + 1) = M(k) + 1 + M(k) = \ldots = 2^{k+1} - 1$.

We have actually constructed the sequence $M(k)$, $k = 1, 2, \ldots$, by *recursion*,

$$\begin{aligned} M(1) &= 1 \\ M(k+1) &= 2M(k) + 1, k = 1, 2, \ldots. \end{aligned}$$

Real Numbers and Binary Expansions

The real numbers in $\mathbf{R}$ are identified with points on a horizontal line. For the time being, we will identify a real number x with a *decimal expansion*.

- Every decimal expansion represents a real number x:

$$x = \pm N.d_1 d_2 \ldots,$$
$$d_k \in \{0, 1, ..., 9\}.$$

This is the statement that every infinite series of the form

$$d_1 10^{-1} + d_2 10^{-2} + \ldots, \quad d_k \in \{0, 1, ..., 9\},$$

converges.

Just as well we could identify a real number x with a *binary expansion*.

- Every binary expansion represents a real number x:

$$x = \pm N._{\text{bin}} b_1 b_2 \ldots,$$
$$b_k \in \{0, 1\}.$$

This is the statement that every infinite series of the form

$$b_1 2^{-1} + b_2 2^{-2} + \ldots, \quad b_k \in \{0, 1\},$$

converges.

Construction of the Binary Expansion by Recursion

A demonstration of a construction of the the binary expansion of a real number x on the line, $0 \leq x < 1$, will be presented in class.