MthT 430 Notes Chapter 5 Equivalent Definitions

What is a definition?

From `http://mw1.merriam-webster.com/dictionary/definition`

Definition of definition - Merriam–Webster Online Dictionary

Main Entry: definition

Function: noun

1: an act of determining ...

2a: a statement expressing the essential nature of something

2b: a statement of the meaning of a word or word group or a sign or symbol <dictionary definitions>

Statement 2b seems most appropriate for mathematics.

Think of a **Definition** as being able to interchange

- [Definition] Term[1] (What is being defined)
- [Definition] Description (Details)

Equivalent Definitions of Limit

Definition. *(Actual, p. 96)*

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Thus we are able to use interchangeably the phrases

- (*Definition Term*) $\lim_{x \to a} f(x) = L$.
- (*Definition Description*) For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

[1] I borrow the words *Definition Term* and *Definition Description* from the `html` tags <`DT`> and <`DD`>.

We wish to decide which variations of a definition are "correct" and give an equivalent definition.

Think of a *Equivalent Definitions* as an *If and Only If Theorem*.

The phrase "**Definition X** is *equivalent* to **Definition Y**" means you can use interchangeably the phrases

- [Definition] Term (What is being defined)
- [Definition] Description X (Details)
- [Definition] Description Y (Details)

To show that two definitions **X** and **Y** for *the same Definition Term* are equivalent we must show the following:

- Satisfying Definition Description X $\Rightarrow$ Satisfying Definition Description Y.
- Satisfying Definition Description Y $\Rightarrow$ Satisfying Definition Description X.

Now if **Definition X** is *not equivalent* to **Definition Y** for *the same Definition Term*, then *at least one* of the following is false:

- Satisfying Definition Description X $\Rightarrow$ Satisfying Definition Description Y.
- Satisfying Definition Description Y $\Rightarrow$ Satisfying Definition Description X.

Interpreting each of the above as a *Theorem*, the way to show a *Theorem* is false is to construct a *counterexample*. A *counterexample* is an object [construct, ...] which satisfies the hypotheses of the proposed Theorem, but does not satisfy the conclusion[s] of the proposed Theorem.

Actual Definition of Limit

Definition ACTUAL. *(Actual, p. 96)*

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Proposed Variations

For each of the proposed variations AA – OO of the actual (Spivak) definition description of

$$\lim_{x \to a} f(x) = L,$$

decide whether the proposed variation **Definition XX** is equivalent to **Definition Actual**. Thus for each you must think about the validity of the the two Theorems:

- Satisfying Definition Description XX $\Rightarrow$ Satisfying Definition Description ACTUAL. If False, there is a counterexample.
- Satisfying Definition Description ACTUAL $\Rightarrow$ Satisfying Definition Description XX. If False, there is a counterexample.

You may construct any counterexample graphically, by formula, or by a precise description.

Definition AA.

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, $0 < |x - a| < \delta$, *and* $|f(x) - L| < \epsilon$.

Definition BB.

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for some* x, $0 < |x - a| < \delta$, *and* $|f(x) - L| < \epsilon$.

Definition CC.

$$\lim_{x \to a} f(x) = L$$

means: For an $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Definition DD.

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$.

Definition EE.

$$\lim_{x \to a} f(x) = L$$

means: For any $\epsilon > 0$, *there is a* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Definition FF.

$$\lim_{x \to a} f(x) = L$$

means: For any $\epsilon > 0$*, there is a* $\delta > 0$ *such that, for all* x*,* $0 < |x - a| < \delta$ *implies* $|f(x) - L| < \epsilon$*.*

Definition GG.

$$\lim_{x \to a} f(x) = L$$

means: For any $\epsilon > 0$*, there is a* $\delta > 0$ *such that, for all* x*,* $|f(x) - L| < \epsilon$ *if* $0 < |x - a| < \delta$*.*

Definition HH.

$$\lim_{x \to a} f(x) = L$$

means: For any $\epsilon > 0$*, there is a* $\delta > 0$ *such that, for all* x*,* $|f(x) - L| < \epsilon$ *and* $0 < |x - a| < \delta$*.*

Definition II.

$$\lim_{x \to a} f(x) = L$$

means: For any $\epsilon > 0$*, there is a* $\delta > 0$ *such that, for all* x*,* $|f(x) - L| < \epsilon$ *whenever* $0 < |x - a| < \delta$*.*

Definition JJ.

$$\lim_{x\to a} f(x) = L$$

means: For every $\epsilon > 0$, there is a $\delta > 0$ such that, for all x, $|f(x) - L| < \epsilon$ for $0 < |x - a| < \delta$.

Definition KK.

$$\lim_{x\to a} f(x) = L$$

means: For an $\epsilon > 0$, there is a $\delta > 0$ such that, for all x, $|f(x) - L| < \epsilon$ for $0 < |x - a| < \delta$.

Definition LL.

$$\lim_{x\to a} f(x) = L$$

means: For a $\delta > 0$, there is an $\epsilon > 0$ such that, for all x, $|f(x) - L| < \epsilon$ provided that $0 < |x - a| < \delta$.

Definition MM.

$$\lim_{x\to a} f(x) = L$$

means: For all $\delta > 0$, there is an $\epsilon > 0$ such that, for all x, $|f(x) - L| < \epsilon$ for $0 < |x - a| < \delta$.

Definition NN.

$$\lim_{x\to a} f(x) = L$$

means: For some $\delta > 0$, there is an $\epsilon > 0$ such that, for all x, if $|f(x) - L| < \epsilon$, then $0 < |x - a| < \delta$.

Definition OO.

$$\lim_{x\to a} f(x) = L$$

means: For some $\delta > 0$, for all $\epsilon > 0$, for all x, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.