MthT 430 Projects Chap 7b – Three Doubtful Theorems if Numbers = Q

While working on Chap7bproj, assume that we are in the *rational world* – the only numbers available are the rational numbers **Q** – there are no irrational numbers!

We could still define limits and continuity for functions, e.g.,

$\epsilon - \delta$ **Definition.** *The function f is continuous at a means: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all rational x, if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*[1]

We could also define continuity on intervals (of course the end points of closed intervals $[a, b]$ are rational).

We investigate the validity of the Three Hard Theorems in this context.

For these projects, assume that we are in the *rational world*. You are allowed to construct functions by formulas so that for all rational numbers, x, $f(x)$ is a rational number. The usual polynomial and rational functions with integer or rational coefficients will be continuous at all points where we do not try to divide by 0. It is a deep result trigonometric functions, etc., are not allowed.

Continuous Functions on Intervals Have the Intermediate Value Property

Doubtful Theorem 1. *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$.*

1. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$,
- $f(0) < 0 < f(1)$,
- There is no $x \in [0, 1]$ such that $f(x) = 0$.

Hint: Use a variation of $x \longrightarrow x^2 - 2$.

[1] Of course for now a, ϵ, δ, and x are all in **Q**.

Continuous Functions on Closed Intervals are Bounded

Doubtful Theorem 2. *If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that $f(x) \leq N$ for all x in $[a, b]$.*

2. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$,
- $f(0) < 0 < f(1)$,
- f is not bounded on $[0, 1]$.

Continuous Functions on Closed Intervals assume a Maximum Value for the Interval

Doubtful Theorem 3. *If f is continuous on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$.*

3. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$,
- f is bounded on $[0, 1]$,
- There is no number y in $[0, 1]$ such that $f(y) \geq f(x)$ for all x in $[0, 1]$.

4. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$,
- $f(0) < 0 < f(1)$,
- There is no number y in $[0, 1]$ such that $f(y) \geq f(x)$ for all x in $[0, 1]$,
- There is no number y in $[0, 1]$ such that $f(y) \leq f(x)$ for all x in $[0, 1]$.