MthT 430 Notes Chap 7d Bolzano – Weierstraß Theorem

For this discussion, we shall assume:

(P13–BIN) Binary Expansions Converge. *Every binary expansion represents a real number x: every infinite series of the form*

$$c_1 2^{-1} + c_2 2^{-2} + \ldots + c_k 2^{-k} + \ldots, \quad c_k \in \{0, 1\},$$

converges to a real number x in $[0, 1]$ and write the binary expansion of x as

$$x = \cdot_{\text{bin}} c_1 c_2 \ldots .$$

In `./chap7b.pdf#BW`, there was an indication of the proof of

The Bolzano–Weierstraß Theorem

Theorem (Bolzano–Weierstraß). *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in. $[0, 1]$. Then there is an x in $[0, 1]$ which is a limit point*[3] *of the sequence $\{x_n\}_{n=1}^{\infty}$.*

The proof will construct a binary expansion for x.

Now – either infinitely many terms of the sequence are 1, in which case $x = 1 = 1\cdot_{\text{bin}}0$ is the desired limit point OR

Ask the question: For infinitely many k, is it true that $x_k \in \left[0, \frac{1}{2^1}\right)$?

If YES, let

$$\begin{aligned} c_1 &= 0, \\ a_1 &= 0 = 0\cdot_{\text{bin}}0, \\ b_1 &= \frac{1}{2} \\ &= a_1 + \frac{1}{2^1}. \\ s_1 &= a_1. \end{aligned}$$

Then

$$b_1 - a_1 = \frac{1}{2^1},$$

Infinitely many x_k are in $[a_1, b_1)$.

[3] A point x is a limit point of the sequence if for every $\epsilon > 0$, infinitely many terms of the sequence are within ϵ of x. Alternately, there is a subsequence which converges to x. A more informal idea is to say that infinitely many terms are as close as desired to x.

If NO, let

$$\begin{aligned} c_1 &= 1, \\ a_1 &= \frac{1}{2} = 0._{\text{bin}}1, \\ b_1 &= 1 = 1._{\text{bin}}0 \\ &= a_1 + \frac{1}{2^1}. \\ s_1 &= a_1. \end{aligned}$$

Then

$$b_1 - a_1 = \frac{1}{2^1},$$

Infinitely many x_k are in $[a_1, b_1)$.

Now continue, ...

$$\begin{aligned} x &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} \left(s_n + \frac{1}{2^n}\right) \end{aligned}$$

Note that $0 \leq x - s_n = |x - s_n| \leq \dfrac{1}{2^n}$.

Fuller Proof of Bolzano–Weierstraß)

Let's be more precise.

Theorem (Bolzano–Weierstraß). *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in. $[0, 1]$. Then there is an x in $[0, 1]$ which is a limit point[3] of the sequence $\{x_n\}_{n=1}^{\infty}$.*

The proof will construct a binary expansion for x.

Now – either infinitely many terms of the sequence are as close as desired to 1, in which case $x = 1 = 1._{\text{bin}}0$ is the desired limit point.

Otherwise –

We will try to construct a sequence $c_k \in \{0, 1\}$ such that if

$$\begin{aligned} a_k &= ._{\text{bin}}c_1 \ldots c_k, \\ b_k &= a_k + \frac{1}{2^k}, \\ I_k &= [a_k, b_k)\,, \end{aligned}$$

[3] A point x is a limit point of the sequence if for every $\epsilon > 0$, infinitely many terms of the sequence are within ϵ of x. Alternately, there is a subsequence which converges to x. A more informal idea is to say that infinitely many terms are as close as desired to x.

then I_k is a decreasing sequence of intervals such that for every k, I_k contains infinitely many terms of the sequence $\{x_n\}$.

Let

$$a_0 = 0,$$
$$b_0 = 1 = a_0 + \frac{1}{2^0},$$
$$I_0 = [a_0, b_0)\,;$$

I_0 contains infinitely many terms of the sequence $\{x_n\}$.

Divide the interval I_0 into two halves, $\left[a_0, a_0 + \frac{1}{2^1}\right)$ and $\left[a_0 + \frac{1}{2^1}, b_0\right)$ – call them the *left half* and *right half* – at least one contains infinitely many terms of the sequence $\{x_n\}$. Pick it (in case of a tie, choose either as you desire). If the *left* is chosen, let

$$c_1 = 0,$$
$$a_1 = a_0 = \cdot_{\text{bin}} c_1,$$
$$b_1 = a_0 + \frac{1}{2^1} = a_1 + \frac{1}{2^1},$$
$$I_1 = [a_1, b_1)\,.$$

If the *right* is chosen, let

$$c_1 = 1,$$
$$a_1 = a_0 + \frac{1}{2^1} = \cdot_{\text{bin}} c_1,$$
$$b_1 = b_0 = a_1 + \frac{1}{2^1},$$
$$I_1 = [a_1, b_1)\,.$$

I_1 contains infinitely many terms of the sequence $\{x_n\}$.

Now continue, by recursion to choose c_k.

If $c_1, \ldots . c_k$, have been chosen so that for $1 \le j \le k$,

$$a_j = \cdot_{\text{bin}} c_1 \ldots c_j,$$
$$b_j = a_j + \frac{1}{2^j},$$
$$I_j = [a_j, b_j)\,,$$

$I_1 \supseteq \ldots \supseteq I_k$, and for $1 \le j \le k$, I_j contains infinitely many terms of the sequence $\{x_n\}$ –

Divide the interval I_k into two halves, $\left[a_k, a_k + \frac{1}{2^{k+1}}\right)$ and $\left[a_k + \frac{1}{2^{k+1}}, b_k\right)$ – call them the *left half* and *right half* – at least one contains infinitely many terms of the sequence

$\{x_n\}$. Pick it (in case of a tie, choose either as you desire). If the *left* is chosen, let

$$\begin{aligned}
c_{k+1} &= 0, \\
a_{k+1} &= a_k = \cdot_{\text{bin}} c_1 \ldots c_{k+1}, \\
b_{k+1} &= a_k + \frac{1}{2^{k+1}} = a_{k+1} + \frac{1}{2^{k+1}}, \\
I_{k+1} &= [a_{k+1}, b_{k+1}) .
\end{aligned}$$

If the *right* is chosen, let

$$\begin{aligned}
c_{k+1} &= 1, \\
a_{k+1} &= a_k + \frac{1}{2^{k+1}} = \cdot_{\text{bin}} c_1 \ldots c_{k+1}, \\
b_{k+1} &= b_k = a_{k+1} + \frac{1}{2^{k+1}}, \\
I_{k+1} &= [a_{k+1}, b_{k+1}) .
\end{aligned}$$

I_{k+1} contains infinitely many terms of the sequence $\{x_n\}$.

The desired limit point x is

$$\begin{aligned}
x &= \lim_{k\to\infty} a_k \\
&= \lim_{k\to\infty} (b_k)
\end{aligned}$$

Note that $0 \leq x - a_k = |x - a_k| \leq \dfrac{1}{2^k}$.