MthT 430 Notes Chap8a Least Upper Bounds and Binary Expansions
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Bounds and Least Upper Bounds

Definition. *A set A of real numbers is* **bounded above** *if there is a number x such that*

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** *for A.*

Definition. *A number x is a* **least upper bound** *for a set A if*

$$x \text{ is an upper bound for } A, \tag{1}$$
$$\text{if } y \text{ is an upper bound for } A, \text{ then } x \leq y \tag{2}$$

Such a number x is also called the **supremum** *for A and sometimes denoted by* $\sup A$ *or* $\operatorname{lub} A$.

Definition. *A number x is a* **greatest element** *of a non empty set A if*

$$\begin{cases} x \in A, \\ x \text{ is an upper bound for } A. \end{cases}$$

Examples

- The set $[0, 1]$ has least upper bound 1.

  $$\sup\,[0, 1] = 1.$$

  The number 1 is also the greatest element of $[0, 1]$.

- If a nonempty set A has a greatest element a_{max}, then

  $$\sup A = a_{\text{max}}.$$

- Let $A = (0, 1)$. Then

  $$\sup A = 1.$$

  The set A has no greatest element.

- Let

  $$\begin{aligned} A &= \left\{x \in \mathbf{Q} \mid \mathbf{x^2 < 2}\right\} \\ &= \left\{x \in \mathbf{Q} \mid \mathbf{x^2 \leq 2}\right\}. \end{aligned}$$

  Then A has no greatest element.

Least Upper Bound Property for Real Numbers R

The following property of real numbers *cannot be proved* from (P1) – (P12).

(P13) or (P13–LUB) Least Upper Bound Property. *If A is a non empty set of real numbers, and A is bounded above, then A has a least upper bound.*

N.B. *A has a least upper bound* as to be interpreted as *there is a number x such that x is a least upper bound for A.*

Note that (P13) does not hold if the only numbers available were **Q**, the rational numbers. There is no *rational number* which is the least upper bound of the set

$$A_{\sqrt{2}} = \left\{ x \mid x \in \mathbf{Q} \text{ and } x^2 < 2 \right\}.$$

Relation between Least Upper Bound and Binary Expansion

Our starting point is that every binary expansion represents a real number. As we said in `MthT 430 Notes Chapter 6a Binary Expansions and Arguments`

- Every binary expansion represents a real number x:

$$\begin{aligned} x &= \pm N._{\text{bin}} c_1 c_2 \ldots, \\ c_k &\in \{0, 1\}. \end{aligned}$$

This is the statement that every infinite series of the form

$$c_1 2^{-1} + c_2 2^{-2} + \ldots, \quad c_k \in \{0, 1\},$$

converges.

Constructing the Binary Expansion of a Least Upper Bound

We give a *folding string argument* to find the *binary expansion* of $\sup A$.

Let A be a nonempty set of real numbers which is bounded above. Without loss of generality (WLG), we assume that A is contained in $[0, 1) \equiv [a_0, b_0)$. Then

$$\text{There is a an upper bound } 1 = b_0 \text{ for } A, \tag{1}$$

$$\text{There is a an } x_0 \in A,\ 0 = a_0 \le x_0 \le 1 = b_0. \tag{2}$$

Now we have that $\hat{y} = \sup A$, if it exists, satisfies $0 = 0._{\text{bin}}0 \le x_0 \le \hat{y} \le b_0 = 1._{\text{bin}}0$.

If $x_0 = b_0$, STOP!

$$\sup A = b_0.$$

A has a greatest element b_0.

If $x_0 < b_0$, we divide the interval $[a_0, b_0)$ into two intervals $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$.

Ask the question: Is $m_0 = (a_0 + b_0)/2$ an upper bound for A?

If the answer is YES, m_0 is an upper bound for A, select the left interval $\left[0, \frac{1}{2}\right) = [a_1, b_1)$ by

$$a_1 = 0 = a_0,$$
$$b_1 = m_0 = \frac{1}{2^1} = a_1 + \frac{1}{2^1}.$$

Then b_1 is an upper bound for A. Let $c_1 = 0$ so that

$$a_1 = \cdot_{\text{bin}} c_1,$$
$$b_1 = a_1 + \frac{1}{2^1}.$$

If the answer is NO, there an $x_1 \in A$, such that $m_0 = 0._{\text{bin}} 1 < x_1 \leq b_0 = 1$.

Select the right interval $\left[\frac{1}{2}, 1\right) = [a_1, b_1)$ by

$$a_1 = m_0,$$
$$b_1 = b_0.$$

Then b_1 is an upper bound for A. Let $c_1 = 1$ so that

$$a_1 = \cdot_{\text{bin}} c_1,$$
$$b_1 = a_1 + \frac{1}{2^1}.$$

In both cases we have constructed an interval $[a_1, b_1)$ such that

$$a_1 = 0._{\text{bin}} c_1,$$
$$b_1 = a_1 + \frac{1}{2^1},$$

such that

$$\begin{cases} a_1 \leq \text{any upper bound for } A, \\ b_1 \text{ is an upper bound for } A. \end{cases}$$

Now suppose that $c_1, \ldots, c_k$, $a_k = 0._{\text{bin}} c_1 \ldots c_k$, $b_1, \ldots, b_k$ have been chosen so that

$$a_k = 0._{\text{bin}} c_1 \ldots c_k,$$
$$b_k = a_k + \frac{1}{2^k},$$

and

$$\begin{cases} a_k \leq \text{any upper bound for } A, \\ b_k \text{ is an upper bound for } A. \end{cases}$$

If $b_k \in A$, STOP!

$$\sup A = b_k.$$

A has a greatest element b_k.

If $b_k \notin A$, divide the interval $[a_k, b_k)$ into two parts by taking the midpoint

$$\begin{aligned} m_k &= \frac{a_k + b_k}{2} \\ &= a_k + \frac{1}{2^{k+1}}. \end{aligned}$$

Ask the question: Is m_k an upper bound for A?

If YES, select the left interval $[a_k, m_k)$ by defining

$$\begin{aligned} c_{k+1} &= 0, \\ a_{k+1} &= a_k \\ &= 0._{\text{bin}} c_1 \ldots c_k c_{k+1}, \\ b_{k+1} &= m_k \\ &= a_{k+1} + \frac{1}{2^{k+1}}. \end{aligned}$$

If NO, select the right interval $[m_k, b_k)$ by defining

$$\begin{aligned} c_{k+1} &= 1, \\ a_{k+1} &= m_k \\ &= 0._{\text{bin}} c_1 \ldots c_k c_{k+1} \\ &= a_k + \frac{1}{2^{k+1}}, \\ b_{k+1} &= b_k \\ &= a_{k+1} + \frac{1}{2^{k+1}}. \end{aligned}$$

In both cases $c_1, \ldots, c_k, c_{k+1}$, $a_1, \ldots, a_k, a_{k+1}$, $b_1, \ldots, b_k, b_{k+1}$ have been chosen so that

$$a_{k+1} = 0._{\text{bin}} c_1 \ldots c_k c_{k+1},$$
$$b_{k+1} = a_{k+1} + \frac{1}{2^{k+1}},$$

and

$$\begin{cases} a_{k+1} \leq \text{any upper bound for } A, \\ b_{k+1} \text{ is an upper bound for } A. \end{cases}$$

Let

$$\begin{aligned} s &= \lim_{k\to\infty} a_k \\ &= 0._{\text{bin}} c_1 c_2 \ldots \\ &= \lim_{k\to\infty} b_k. \end{aligned}$$

Then

- s is an upper bound for A. Why? Fix $x \in A$. Then for all k, $x \leq b_k$, and so $x \leq \lim_{k\to\infty} b_k = s$.
- $s = \sup A$. Why?

N.B. Even if a system of numbers (such as $\mathbf{Q}$), satisfies (P1 – P12), without assuming P13–LUB, the above construction of the sequences $c_1, \ldots$, $a_1, \ldots$, and $b_1, \ldots$, can be accomplished. So – if, in fact, *all binary expansions converge to a number in the system*, the supremum of the set A has been constructed. We refer to the property *all binary expansions converge to a number in the system* as (P-13–BIN).

BISHL: Bounded Increasing Sequences Have Limits

A consequence of (P13–LUB) is that *Bounded Increasing Sequences Have Limits*.

Theorem. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded monotone nondecreasing sequence; i.e.*

$$x_1 \leq x_2 \leq \ldots,$$

and there is a number M such that for $n = 1, 2, \ldots$,

$$x_n \leq M.$$

Then there is a number L such that

$$\lim_{n\to\infty} x_n = L.$$

Proof using (P13–LUB): Try

$$L = \sup_n x_n.$$

For all n, $x_n \leq L$ (L is an upper bound). Given $\epsilon > 0$, there is a natural number N such that $L - \epsilon < x_N \leq N$ ($L - \epsilon$ is not an upper bound!). Then for all $n \geq N$,

$$L - \epsilon < x_n \leq L,$$

or

$$|x_n - L| = L - x_n < \epsilon.$$

Convergence (Meaning) of Binary Expansions implies (P13–LUB)

Let us state a variation of (P13–LUB).

(P13–BIN) – Binary Expansions Converge. *Every binary expansion represents a real number x: every infinite series of the form*

$$c_1 2^{-1} + c_2 2^{-2} + \ldots, \quad c_k \in \{0, 1\},$$

converges to a real number x, $0 \leq x \leq 1$.

We write

$$x = \pm N._{\text{bin}} c_1 c_2 \ldots, \\ c_k \in \{0, 1\}.$$

Another way to say this is that every infinite series of the form

$$\sum_{k=1}^{\infty} c_k 2^{-k}, \quad c_k \in \{0, 1\},$$

converges to a real number x:

$$x = \lim_{N \to \infty} \sum_{k=1}^{N} c_k 2^{-k}.$$

(P13–BIN) Implies (P13–LUB)

We have shown: If the real numbers (or any number system!) satisfies (P1 – P12) and (P13–BIN), then this set of numbers satisfies (P1 – P12) and (P13–LUB).

LUB2BIN: (P13–LUB) Implies (P13–BIN)

The *converse* is also true: If the real numbers (or any number system!) satisfies (P1 – P12) and (P13–LUB), then this set of numbers satisfies (P1 – P12) and (P13–BIN). This would follow if we could show that the nondecreasing sequence (partial sums)

$$s_N = \sum_{k=1}^{N} c_k 2^{-k}, \quad c_k \in \{0, 1\},$$

is *bounded above*. Then

$$\begin{aligned} x &= \sup_N \sum_{k=1}^{N} c_k 2^{-k} \\ &= \lim_{N \to \infty} \sum_{k=1}^{N} c_k 2^{-k} \\ &= \sum_{k=1}^{\infty} c_k 2^{-k}. \end{aligned}$$

To show that the sequence of partial sums $\{s_N\}$ is *bounded above*, we begin with a BARE HANDS calculation on a geometric series.

Lemma: Geometric Series – BARE HANDS. *If* $|r| < 1$,

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}, \tag{♣}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}. \tag{♠}$$

Proof: To show (♣),

$$(1 - r)\left(1 + \ldots + r^N\right) = 1 - r^{N+1},$$

and for $1 - r \neq 0$, divide by $1 - r$.

To show (♠), if $|r| < 1$,

$$\lim_{N\to\infty} r^{N+1} = 0,$$

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r}.$$

$$= \frac{1}{1 - r}.$$

Theorem. *Property (P13–LUB) implies Property (P13–BIN).*

Proof: Assuming Property (P13–LUB), and of course Properties (P1 – P12), we show that the non decreasing sequence

$$s_N = \sum_{k=1}^{N} c_k 2^{-k}, \quad c_k \in \{0, 1\},$$

is bounded above:

$$\begin{aligned}\sum_{k=1}^{N} c_k 2^{-k} &\leq \sum_{k=1}^{N} 1 \cdot 2^{-k} \\ &\leq \sum_{k=1}^{\infty} 1 \cdot 2^{-k} \\ &= \frac{1}{1 - \frac{1}{2}} - 1 \\ &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty} c_k 2^{-k} &= \lim_{N\to\infty} \sum_{k=1}^{N} c_k 2^{-k} \\ &= \sup_N \sum_{k=1}^{N} c_k 2^{-k} \\ &\leq 1.\end{aligned}$$