MthT 430 Notes Chapter 8b Least Upper Bounds – Equivalent Statements

Assuming (P1 – P12), there are several equivalent statements of the Least Upper Bound Property.

(P13 or P13–LUB) – Least Upper Bound Property. *If A is a non empty set of real numbers, and A is bounded above, then A has a least upper bound.*

(P13–BIN) Binary Expansions Converge. *Every binary expansion represents a real number x: every infinite series of the form*

$$c_1 2^{-1} + c_2 2^{-2} + \dots, \quad c_k \in \{0, 1\},$$

converges to a real number x, $0 \le x \le 1$.

(P13–DECIMALS)– Decimal Expansions Converge. *Every decimal expansion represents a real number x: every infinite series of the form*

$$c_1 10^{-1} + c_2 10^{-2} + \dots, \quad c_k \in \{0, \dots, 9\},$$

converges to a real number x, $0 \le x \le 1$.

The equivalence of (P13) and (P13–BIN) is shown in chap8a.tex.

See `http://www.math.uic.edu/~jlewis/mtht430/chap8a.pdf#BIN`

(P13–BISHL) – Bounded Increasing Sequences Have Limits. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded monotone increasing sequence; i.e.*

$$x_1 \le x_2 \le \dots,$$

and there is a number M such that for $n = 1, 2, \dots$,

$$x_n \le M.$$

Then there is a number L such that

$$\lim_{n\to\infty} x_n = L.$$

(P13–BIN) implies (P13–BISHL) is shown in chap7c.tex.

See `http://www.math.uic.edu/~jlewis/mtht430/chap7c.pdf#BISHL`

(P13–BW) – Bolzano–Weierstraß Property. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of points in.* $[0, 1]$. *Then there is an* x *in* $[0, 1]$ *which is a limit point*[1] *of the sequence* $\{x_n\}_{n=1}^{\infty}$.

(P13–BIN) implies (P13–BW) was shown in chap7b.tex.

See http://www.math.uic.edu/˜jlewis/mtht430/chap7b.pdf#BW

Other Statements Equivalent to (P13–LUB)

Assuming (P1 – P12), there are other statements equivalent to (P13-LUB):

(P13–CFIVP) – Continuous Functions on Intervals Have the Intermediate Value Property. *If* f *is continuous on* $[a, b]$ *and* $f(a) < 0 < f(b)$, *then there is some* x *in* $[a, b]$ *such that* $f(x) = 0$.

(P13–BIN) implies (P13–CFIVP) was shown in chap7b.tex.

See http://www.math.uic.edu/˜jlewis/mtht430/chap7b.pdf#CFIVP

(P13–CFCIB) – Continuous Functions on Closed Intervals are Bounded. *If* f *is continuous on* $[a, b]$, *then* f *is bounded above on* $[a, b]$, *that is, there is some number* N *such that* $f(x) \leq N$ *for all* x *in* $[a, b]$.

(P13–CFCIMAX) – Continuous Functions on Closed Intervals assume a Maximum Value for the Interval. *If* f *is continuous on* $[a, b]$, *then there is a number* y *in* $[a, b]$ *such that* $f(y) \geq f(x)$ *for all* x *in* $[a, b]$.

(P13–HeineBorel) – Heine–Borel Theorem. *Every open cover of a closed interval contains a finite subcover of the closed interval.*

(P13–CAUCHY) Cauchy Sequences Have Limits. *If* $\{x_n\}$ *is a Cauchy sequence*[2], *then there is a number* x *such that*

$$\lim_{n\to\infty} x_n = x.$$

This property is often stated: The real numbers are complete.

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[1] A point x is a limit point of the sequence if for every $\epsilon > 0$, infinitely many terms of the sequence are within ϵ of x. Alternately, there is a subsequence which converges to x. A more informal idea is to say that infinitely many terms are as close as desired to x.

[2] Look up the definition of Cauchy sequence. A working definition given by Konrad Knopp in **Introduction to the Theory of Functions** is that *almost all the terms are close together*.

(P13–CFCIUC) Continuous Functions on Closed Bounded Intervals are Uniformly Continuous. *If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$. See Spivak, p. 143.*

(P13–CONNECTED) Closed Intervals are Connected[3].

[3] Look up the definition of a connected set.