MthT 430 Projects Chap 8b – Intermediate Value Property

1. Let f be a continuous function on $[0, 1]$ such that
   - $f(0) > 0$,
   - $f(1) < 1$.

   Draw a graph of several (not too complicated) continuous functions, f, satisfying the two properties to decide whether there is always an x, $0 < x < 1$, such that $f(x) = x$.

   Now prove that there is an x, $0 < x < 1$, such that $f(x) = x$.

   See also Spivak, Chapter 7, Problem 11.

2. Suppose that
   - f and g are continuous functions on $[0, 1]$
   - $f(0) > g(0)$,
   - $f(1) < g(1)$.

   Draw graphs of several pairs of continuous functions, f, g, satisfying the three properties to decide whether there is always an x, $0 < x < 1$, such that $f(x) = g(x)$.

   Now prove that if f, g, are continuous there is an x, $0 < x < 1$, such that $f(x) = g(x)$.

3. Suppose we are working with a **number system** (such as the rational numbers **Q**) which satisfies (P1 – P12), but does **not** satisfy (P13–LUB); *id est*, there is a non empty set A of numbers, A is bounded above, but A does not have a least upper bound. For this A, let

   $$B_A \equiv \{b \mid b \quad \text{is an upper bound for } A.\}$$

   Define

   $$f(x) = \begin{cases} 1, & x \in B_A, \\ -1, & x \notin B_A. \end{cases}$$

   Show that f is continuous at all x, but does **not** satisfy the Intermediate Value Property (IVP).

   Thus NOT (P13–LUB) implies NOT (P13–CFIVP). This shows that (P13–CFIVP) implies (P13–LUB) as stated in chap8b.tex.

   See `http://www.math.uic.edu/~lewis/mtht430/chap8b.pdf#CFIVP`