MthT 430 Projects Chap 8f Note – $\limsup$ and $\liminf$

Really Understanding $\sup$ and $\inf$

- (See also Spivak Chapter 8 - Problem 18) Let $\{x_k\}$ be a bounded sequence. We define the *limit superior* and *limit inferior* of the sequence to be

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty} \left(\sup_{n\geq k} x_n \right),$$
$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty} \left(\inf_{n\geq k} x_n \right).$$

- (P13–BISHL) shows that both $\limsup\limits_{k\to\infty} x_k$ and $\liminf\limits_{k\to\infty} x_k$ exist.

In `http://www.math.uic.edu/~lewis/mtht430/2007project.pdf`, we made the definition

Definition. *If f is a bounded function on $[0,1]$, we define*

$$\sup f = \sup_{x\in[0,1]} f(x),$$
$$\inf f = \inf_{x\in[0,1]} f(x).$$

We could just replace the domain of the function f, $[0,1]$, by a set of natural numbers $Z_k = \{k, k+1, \ldots\}$, and use the usual conventions for sequences to make the definition:

Definition. *If $\{x_k\}$ is a bounded sequence of real numbers, we define*

$$\overline{x}_k = \sup_{n\geq k} x_n,$$
$$\underline{x}_k = \inf_{n\geq k} x_n.$$

Note that $\{\overline{x}_k\}$ is a nonincreasing sequence which is bounded below so that

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty} \overline{x}_k = \inf_k \overline{x}_k$$

exists. Similarly $\{\underline{x}_k\}$ is a nondecreasing sequence which is bounded above so that

$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty} \underline{x}_k = \sup_k \underline{x}_k$$

exists.

N.B. For every k,

$$\underline{x}_k \leq \overline{x}_k.$$

It follows that

$$\liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k$$

Necessary and Sufficient Condition (NASC) for Existence of a Limit of a Sequence

- Show that
$$\limsup_{k\to\infty} x_k = A$$
if and only if for every $\epsilon > 0$,
$$\begin{cases} x_k > A + \epsilon & \text{for at most finitely many } k, \\ x_k > A - \epsilon & \text{for infinitely many } k. \end{cases}$$

- Show that
$$\liminf_{k\to\infty} x_k = A$$
if and only if for every $\epsilon > 0$,
$$\begin{cases} x_k < A - \epsilon & \text{for at most finitely many } k, \\ x_k < A + \epsilon & \text{for infinitely many } k. \end{cases}$$

- Prove:

Theorem. *Let $\{x_k\}$ be a bounded sequence. Then*

$$\lim_{k\to\infty} x_k \text{ exists}$$

if and only if

$$\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k.$$

Proof. If $\lim_{k\to\infty} x_k = L$, show that $\liminf_{k\to\infty} x_k = \lim_{k\to\infty} x_k.$, ... For the converse, assume $\liminf_{k\to\infty} x_k = A = \limsup_{k\to\infty} x_k$. Given $\epsilon > 0$, except for at most finitely many k, $A - \epsilon < x_k < A + \epsilon$.

Inequalities with $\limsup$ **and** $\liminf$

See `http://www.math.uic.edu/~lewis/mtht430/2007project.pdf`

N.B. Let $\{x_k\}$ and $\{y_k\}$ be bounded sequences. Following the proof of Problem 10 in `2007project.pdf`, for each k have

$$\begin{aligned}\inf_{n\geq k} x_n + \inf_{n\geq k} y_n &\leq \inf_{n\geq k}(x_n+y_n)\\ &\leq \inf_{n\geq k} x_n + \sup_{n\geq k} y_n\\ &\leq \sup_{n\geq k}(x_n+y_n)\\ &\leq \sup_{n\geq k} x_n + \sup_{n\geq k} y_n.\end{aligned}$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

$$\limsup_{k\to\infty}(x_k+y_k) \leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Give an example of bounded sequences $\{x_k\}$ and $\{y_k\}$ such that

$$\limsup_{k\to\infty}(x_k+y_k) < \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

$$\liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \leq \limsup_{k\to\infty}(x_k+y_k).$$

- The general result is that for two bounded sequences $\{x_k\}$ and $\{y_k\}$,

$$\begin{aligned}\liminf_{k\to\infty} x_k + \liminf_{k\to\infty} y_k &\leq \liminf_{k\to\infty}(x_k+y_k)\\ &\leq \liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k\\ &\leq \limsup_{k\to\infty}(x_k+y_k)\\ &\leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.\end{aligned}$$

- The Monster Counterexample to Equality:

$$\begin{aligned}\{x_k\} &= \{2,2,0,0,2,2,0,0,\ldots\},\\ \{y_k\} &= \{0,1,1,2,0,1,1,2,\ldots\},\\ \{x_k+y_k\} &= \{2,3,1,2,2,3,1,2,\ldots\}\end{aligned}$$