MthT 430 Notes Chapter 8g More Equivalent Statements of (P13)

Assuming (P1 – P12), there are several equivalent statements of the Least Upper Bound Property (P13).

(P13 or P13–LUB) – Least Upper Bound Property. *If A is a non empty set of real numbers, and A is bounded above, then A has a least upper bound.*

(P13–BIN) Binary Expansions Converge. *Every binary expansion represents a real number x: every infinite series of the form*

$$c_1 2^{-1} + c_2 2^{-2} + \ldots, \quad c_k \in \{0, 1\},$$

converges to a real number x, $0 \le x \le 1$.

(P13–DECIMALS)– Decimal Expansions Converge. *Every decimal expansion represents a real number x: every infinite series of the form*

$$c_1 10^{-1} + c_2 10^{-2} + \ldots, \quad c_k \in \{0, \ldots, 9\},$$

converges to a real number x, $0 \le x \le 1$.

The equivalence of (P13) and (P13–BIN) is shown in chap8a.tex.

See `http://www.math.uic.edu/~lewis/mtht430/chap8a.pdf#BIN`

(P13–BISHL) – Bounded Increasing Sequences Have Limits. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded monotone increasing sequence; i.e.*

$$x_1 \le x_2 \le \ldots,$$

and there is a number M such that for $n = 1, 2, \ldots$,

$$x_n \le M.$$

Then there is a number L such that

$$\lim_{n\to\infty} x_n = L.$$

Note that

$$\lim_{n\to\infty} x_n = \sup_n x_n.$$

(P13–BIN) implies (P13–BISHL) is shown in chap7c.tex.

See `http://www.math.uic.edu/~lewis/mtht430/chap7c.pdf#BISHL`

(P13–CFCBIB) – Continuous Functions on Closed Bounded Intervals are Bounded. *If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that $f(x) \le N$ for all x in $[a, b]$.*

CFCBIB2BISHL: (P13–CFCBIB) implies (P13–BISHL) is shown as follows (Thanks to a hint from Brayton Gray):

Proof. The proof is by contradiction. Suppose (P13–BISHL) is false. Then there is a bounded strictly increasing sequence, $\{x_n\}$, which does not have a limit or sup. Let $0 = x_0 < x_1 < x_2 < \ldots$, be bounded above by, say, $\frac{1}{2}$. Let

$$B = \{x \in [0, 1] \,|\, x \text{ is an upper bound for } \{x_n\}\}\,,$$
$$A = [0, 1] \setminus B.$$

Then B and A are both nonempty and open in $[0, 1]$.

For $x \in [0, 1]$, define

$$f(x) = 0, x \in B,$$
$$f(x_k) = 2^k, k = 0, 1, \ldots,$$
$$f(x) = f(x_k) + \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}(x - x_k)\,, \text{linear}, x_k \le x \le x_{k+1}.$$

Then f is a continuous function on $[0, 1]$ which is unbounded and (P13–CFCBIB) is not satisfied.

(P13–BW) – Bolzano–Weierstraß Property. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in. $[0, 1]$. Then there is an x in $[0, 1]$ which is a limit point*[1] *of the sequence $\{x_n\}_{n=1}^{\infty}$.*

(P13–BIN) implies (P13–BW) was shown in chap7b.tex.

See `http://www.math.uic.edu/~lewis/mtht430/chap7b.pdf#BW`

Other Statements Equivalent to (P13)

Assuming (P1 – P12), there are other statements equivalent to (P13):

(P13–CFIVP) – Continuous Functions on Intervals Have the Intermediate Value Property. *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$.*

(P13–BIN) implies (P13–CFIVP) was shown in chap7b.tex.

[1] A point x is a limit point of the sequence if for every $\epsilon > 0$, infinitely many terms of the sequence are within ϵ of x. Alternately, there is a subsequence which converges to x. A more informal idea is to say that infinitely many terms are as close as desired to x.

See http://www.math.uic.edu/~lewis/mtht430/chap7b.pdf#CFIVP

(P13–CFCBIMAX) – Continuous Functions on Closed Intervals assume a Maximum Value for the Interval. *If f is continuous on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$.*

(P13–HB) – Heine–Borel Theorem. *Every open cover of a closed bounded interval contains a finite subcover of the closed interval.*

(P13–CAUCHY) – Cauchy Sequences Have Limits. *If $\{x_n\}$ is a Cauchy sequence*[2]*, then there is a number x such that*

$$\lim_{n \to \infty} x_n = x.$$

This property is often stated: The real numbers are complete.

(P13–CFCIUC) Continuous Functions on Closed Bounded Intervals are Uniformly Continuous. *If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$. See Spivak, p. 143.*

(P13–CONNECTED) – Intervals are Connected. *An open [closed] interval cannot be decomposed into two disjoint nonempty open [closed] subsets.*

With a little bit of topology, (P13–CFIVP) can be shown to be equivalent to (P13–CONNECTED).

[2] Look up the definition of Cauchy sequence. A working def inition given by Konrad Knopp in **Introduction to the Theory of Functions** is that *almost all the terms are close together*.