The Algebra Symposium: Notes on Bouncing Balls

From **Calculus**, Hughes–Hallett, *et al.*:

1. A ball is dropped from a height of 10 feet and bounces. Each bounce is $\frac{3}{4}$ of the height of the bounce before. Thus after the ball hits the floor for the first time, the ball rises to a height of $10\left(\frac{3}{4}\right) = 7.5$ feet, and after the it hits the floor for the second time, the ball rises to a height of $7.5\left(\frac{3}{4}\right) = 10\left(\frac{3}{4}\right)^2 = 5.625$ feet.

(a) Find an expression for the height to which the ball rises after it hits the floor for the n^{th} time.

After it hits the floor for the n^{th} time, the ball rises to a height of $10\left(\frac{3}{4}\right)^n$ feet.

(b) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the first, second, third, and fourth times.

After one time the ball has traveled 10 feet. After two times the ball has traveled $10 + 2 \cdot 10\left(\frac{3}{4}\right)^1$ feet. After three times the ball has traveled $10 + 2 \cdot 10\left(\frac{3}{4}\right)^1 + 2 \cdot 10\left(\frac{3}{4}\right)^2$ feet.

(c) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the n^{th} time. Express your answer in a closed form.

Hint

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

After hitting the floor for the n^{th} time, the next bounce adds $2 \cdot 10\left(\frac{3}{4}\right)^n$ to the total vertical distance. Thus the total vertical distance traveled after hitting for the n^{th} time is

$$10\left(1 + 2 \cdot \left(\frac{3}{4}\right)^1 + \ldots + 2 \cdot \left(\frac{3}{4}\right)^{n-1}\right).$$

A closed form of this expression is

$$10 \cdot \left(1 + 2\sum_{k=1}^{n-1}\left(\frac{3}{4}\right)^k\right) = 10\left(1 + 2\frac{(3/4) - (3/4)^n}{1 - (3/4)}\right)$$

If we let $n \to \infty$, the total vertical distance traveled is

$$10\left(1 + 2\frac{(3/4)}{1 - (3/4)}\right) = 70\text{feet}.$$

2. You might think that the ball [in the previous problem] keeps bouncing forever since it takes infinitely many bounces.

   Is this true?

   Although there are infinitely many bounces, they occur in finite time. If a ball falls from a height h feet, its vertical position at time t is given by

$$y = h - \frac{1}{2}gt^2,$$

where $g \approx 32$ feet/(sec)^2 is the gravitational constant. Thus $y = 0$ when

$$\begin{aligned} t &= \sqrt{2h/g} \\ &= \sqrt{2/g}\sqrt{h} \\ &\approx \frac{1}{4}\sqrt{h}. \end{aligned}$$

An argument similar to the calculation of the total total vertical distance traveled will show that the total time elapsed traveled after hitting for the n^{th} time is

$$10\sqrt{\frac{2}{g}}\left(1 + 2\cdot\left(\sqrt{3/4}\right)^1 + \ldots + 2\cdot\left(\sqrt{3/4}\right)^{n-1}\right).$$

A closed form of this expression is

$$10\sqrt{\frac{2}{g}}\cdot\left(1 + 2\sum_{k=1}^{n-1}\left(\sqrt{3/4}\right)^k\right) = 10\sqrt{\frac{2}{g}}\left(1 + 2\frac{\sqrt{3/4} - \sqrt{3/4}^n}{1 - \sqrt{3/4}}\right)$$

If we let $n \to \infty$, the total time elapsed is

$$10\sqrt{\frac{2}{g}}\left(1 + 2\frac{\sqrt{3/4}}{1 - \sqrt{3/4}}\right) \approx *\text{sec}.$$

We exploit the simultaneous convergence of the two geometric series:

$$\sum_{k=1}^{\infty}(\text{bounce ratio})^k,$$
$$\sum_{k=1}^{\infty}\left(\sqrt{\text{bounce ratio}}\right)^k.$$