The Algebra Symposium: Race Track Comments

From **Math Olympics - Rome**, March 21, 1997:

1. Two cars traveling at constant speed on a track are side by side every 56 minutes. If, with the same speeds, one of the cars were traveling in the opposite direction, the two cars would meet every 8 minutes. How long does it take the faster car to complete one lap on the track?

A Solution

Several variables come to mind:

$$\begin{aligned}
v_f &= \text{speed of fast car (units?)},\\
v_s &= \text{speed of slow car (units?)},\\
T_l &= \text{time taken to lap (given)},\\
T_m &= \text{time taken to meet (given)},\\
T_f &= \text{lap time for the faster car},\\
T_s &= \text{lap time for the slower car},\\
L &= \text{length of one lap.}
\end{aligned}$$

Using that $L = (\text{speed}) \times (\text{lap time})$, we have the two equations

$$\begin{aligned}
(v_f - v_s) \cdot 56 &= L,\\
(v_f + v_s) \cdot 8 &= L.
\end{aligned}$$

The problem asks to find $T_f = \dfrac{L}{v_f}$.

I'm worried – I have two equations for three unknowns. Adding and subtracting 8 times the first and 56 times the second,

$$\begin{aligned}
2 \cdot 56 \cdot 8 \cdot v_f &= (56 + 8)\, L,\\
2 \cdot 56 \cdot 8 \cdot v_s &= (56 - 8)\, L.
\end{aligned}$$

Success!

$$\begin{aligned}
T_f &= \frac{L}{v_f}\\
&= \frac{2 \cdot 56 \cdot 8}{56 + 8}.
\end{aligned}$$

Notes

1. It seems we could determine the lap time of the slower car, T_s.

2. With the given data, we cannot determine the actual speeds of the two cars. What additional data would enable us to determine the actual speeds?

3. If we knew the actual speed of the faster car, e.g., 200 km/hr, could we determine the speed of the slower car?

4. The reason I did not simplify the numbers in my solution was that I wanted develop an *algebraic method* to solve the problem for general data, T_l, and T_m.

Algebraic Method

Using that $L =$ (speed) $\times$ (lap time), we have the two equations

$$\begin{aligned}(v_f - v_s) \cdot T_l &= L,\\ (v_f + v_s) \cdot T_m &= L.\end{aligned}$$

The problem asks to find $T_f = \dfrac{L}{v_f}$.

I'm worried – I have two equations for three unknowns. Adding and subtracting T_m times the first and T_l times the second,

$$\begin{aligned}2 \cdot T_l \cdot T_m \cdot v_f &= (T_l + T_m)\, L,\\ 2 \cdot T_l \cdot T_m \cdot v_s &= (T_l - T_m)\, L.\end{aligned}$$

Success!

$$\begin{aligned}T_f &= \frac{L}{v_f}\\ &= \frac{2 \cdot T_l \cdot T_m}{T_l + T_m}.\end{aligned}$$