Strong minimal sets in differentially closed fields: Equations of Poizat and Lienard type

IPM Logic Seminar

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Outline

1. Review of the basics on the model theory of differential fields.
2. Survey known some of the known results on strongly minimal sets in general and, more specifically, in differentially closed fields.
3. Describe some recent work with Jim Freitag, Rémi Jaoui and Ronnie Nagloo.

Basics

A *differential field* is a field K of characteristic 0 with a derivation $D : K \to K$

$$D(a+b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a).$$

A *differential polynomial* in variables $X_1, \ldots, X_n$ over K is an element of the ring $K\{X_1, \ldots, X_n\}$ which is

$$K[X_1, \ldots, X_n, D(X_1), \ldots, D(X_n), \ldots, D^{(m)}(X_1), \ldots, D^{(m)}(X_n), \ldots].$$

The *order* of $f \in K\{X_1, \ldots, X_n\}$ is the largest m such that some $D^{(m)}$ occurs.

The constant subfield of K is $C(K) = \{x \in K : D(x) = 0\}$.

Prehistory

A differential field K is *existentially closed* if for any finite system Σ of polynomial differential equations having a solution in some $L \supseteq K$ already has a solution in K.

Theorem (Robinson 1959)

The theory of existentially closed differential fields T is axiomatizable, complete, model complete and decidable.

Robinson's axiomatization was quite difficult and built on theory of differential ideals developed by Ritt and Kolchin.

Blum's Thesis 1968

Definition

We say that (K, D) is *differentially closed* $K \models \mathrm{DCF}$ if
- K is an algebraically closed field of characteristic zero;
- If $f(X), g(X) \in K\{X\}$ are nonzero and $\mathrm{order}(f) > \mathrm{order}(g)$, then there is $x \in K$ such that $f(x) = 0 \wedge g(x) \neq 0$.

Theorem (Blum)

DCF is a complete theory with quantifier elimination axiomatizing the theory of existentially closed differential fields.

Examples of DCF?

Until recently they were no "natural" algebraic or analytic examples of differentially closed fields.

Though, results of Seidenberg show that any countable differential field is isomorphic to a field of germs of complex meromorphic functions.

Let K be an algebraically closed field. Let $K\langle\langle t\rangle\rangle$ be the field of *Puiseux* series over K.

$$K\langle\langle t\rangle\rangle = \bigcup_{n=1}^{\infty} K((t^{\frac{1}{n}})).$$

Fact: $K\langle\langle t\rangle\rangle$ is algebraically closed.

For later, $K\langle\langle t\rangle\rangle$ is a valued field under the valuation if $f = \sum_{i=m}^{\infty} a_i t^{i/n}$ where $a_m \neq 0$, then $v(f) = \frac{m}{n}$.

Suppose δ is a derivation on K. We can extend δ to a derivation D on $K\langle\langle t\rangle\rangle$ by

$$D\left(\sum a_i t^{\frac{i}{n}}\right) = \sum \delta(a_i) t^{\frac{i}{n}} + \sum \frac{i}{n} a_i t^{\frac{i}{n}-1}$$

Theorem (Léon-Sánchez, Tressl)

Let K_0 be algebraically closed field with the trivial derivation. Let $K_{n+1} = K_n\langle\langle t_n\rangle\rangle$ with the derivation as above. Then $\bigcup K_n$ is differentially closed.


ω-stability

Let $K \models$ DCF and p be a 1-type over a differential field $k \subseteq K$.
By quantifier elimination p is determined by
$I_p = \{f \in k\{X\} : f(x) = 0 \in p\}$, a prime differential ideal.
$p \mapsto I_p$ is a bijection between $S_1(k)$ and prime differential ideals of $k\{X\}$.

Theorem (Ritt–Raudenbush Basis Theorem)

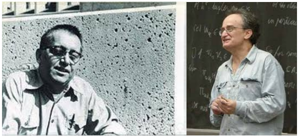
Any radical differential ideal in $k\{X_1, \ldots, X_n\}$ is finitely generated.

Thus $|S_1(k)| = |k|$.

Corollary (Blum)

DCF is ω-stable.

Differential Closures



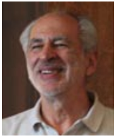
Definition

We say that $K \supseteq k$ is a *differential closure* of k if K is differentially closed and for any differentially closed $L \supseteq k$ there is a differential embedding $j : K \to L$ fixing k pointwise.

Blum observed differential closures = prime model extensions in DCF.

Theorem

In ω-stable theories:
[Morley 1965] *prime model extensions exist;*
[Shelah 1972] *prime model extensions are unique; i..e., if $\mathcal{M}$ and $\mathcal{N}$ are prime over A there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ fixing A.*



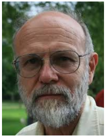
The Least Misleading Example

Gerald Sacks in his 1972 book *Saturated Model Theory* described differentially closed fields as the *least misleading example* of an ω-stable theory.

Reasons to Study DCF

- Many interesting phenomena from pure model theory, particularly geometric stability theory, have found natural manifestations in differentially closed fields.
- Model theoretic methods have provided useful insights into differential algebraic groups, differential algebraic geometry and differential Galois theory.
- Model theoretic and differential algebraic methods have combined in applications to number theory (diophantine and transcendence questions)

Strongly Minimal Sets




Definition

A definable set X is *strongly minimal* if X is infinite and for every definable $Y \subset X$ either Y or $X \setminus Y$ is finite.

General Examples:

- ACF: K an algebraically closed field and $X \subseteq K^n$ an irreducible algebraic curve ($\pm$ finitely many points).
- DCF: $\mathbb{K}$ differentially closed, C the field of constants.
- Equality: $\mathcal{M}$ an infinite set with no structure and $X = \mathcal{M}$.
- Successor: $\mathcal{M}$ an infinite set $f : \mathcal{M} \to \mathcal{M}$ a bijection with no finite orbits, $X = \mathcal{M}$.
- DAG: $\mathcal{M}$ a torsion free divisible abelian group, $X \subseteq \mathcal{M}^n$ a translate of a one-dimensional subspace defined over $\mathbb{Q}$.

Model Theoretic Algebraic Closure

Definition

If $a \in \mathcal{M}$, $B \subset \mathcal{M}$, a is *algebraic* over B if there is a formula $\phi(x, y_1, \ldots, y_m)$ and $\mathbf{b} \in B^m$ such that $\phi(a, \mathbf{b})$ and $\{x \in \mathcal{M} : \phi(x, \mathbf{b})\}$ is finite.
Let $\mathrm{cl}(B) = \{a : a \text{ algebraic over } B\}$.

- ACF: $\mathrm{cl}(A)=$ algebraic closure of field generated by A.
- DCF: $\mathrm{cl}(A)=$algebraic closure of differential field generated by A.
- equality: $\mathrm{cl}(A) = A$.
- Successor: $\mathrm{cl}(A) = \bigcup_{a \in A}$ orbit of a under s and s^{-1}
- DAG: $\mathrm{cl}(A) = \mathrm{span}_{\mathbb{Q}}(\mathrm{A})$.

Geometry of Strongly Minimal Sets

Definition

A strongly minimal set X is *geometrically trivial* or *degenerate* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for all $A \subseteq X$.

Equality and Successor are geometrically trivial

Definition

A strongly minimal set X is *modular* if $c \in \mathrm{cl}(B \cup \{a\})$, then $c \in \mathrm{cl}(b, a)$ for some $b \in cl(B)$, for all $a \in X$, $B \subseteq X$.

DAG is non-trivial modular: If $c = \sum m_i b_i + na$ where $m_i, n \in \mathbb{Q}$, then $c = b + na$ where $b = \sum m_i b_i$.

ACF is non-modular

When are two strongly minimal sets "the same"?

Definition

Two strongly minimal sets X and Y are *non-orthogonal* ($X \not\perp Y$) if there is a definable finite-to-finite correspondence $R \subseteq X \times Y$.

Idea: "non-orthogonal"= intimately related, "orthogonal"=not related.

In ACF: If X is a curve there is $\rho : X \to K$ rational so $X \not\perp K$.

In DCF: If X and Y are strongly minimal sets defined over a differentially closed field K, then $X \perp Y$ if and only if for $\mathbf{a} \in X(\mathbb{K}) \setminus X(K)$, $Y(K\langle \mathbf{a} \rangle^{\mathrm{dcl}}) = Y(K)$–i.e., adding points to X does not force us to add points to Y.

Zilber's Principle

Zilber's Principle: Complexity of the combinatorial geometry is an avatar of algebraic structure.

trivial strongly minimal sets have no infinite definable groups

modular strongly minimal sets are controlled by groups.

Theorem (Hrushovski)

If X is a nontrivial modular strongly minimal set, there is an interpretable modular strongly minimal group G such that $X \not\perp G$.

Zilber Conjectured that non-modular strongly minimal sets only occur in the presence of an algebraically closed field, but Hrushovski refuted this in general.

Strongly Minimal Sets in DCF–Early Results

- The field of constants C is non-locally modular. Indeed if $X \subseteq C^n$ is definable in K, X is definable in $(C, +, \cdot)$.
- There are many trivial strongly minimal sets

Theorem (Kolchin/Rosenlicht/Shelah 1974)

The differential equation $y' = y^3 - y^2$ defines a trivial strongly minimal set. If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are distinct solutions with $a_i, b_i \neq 0, 1$, then there is an automorphism σ of $\mathbb{K}$ with $\sigma(a_i) = b_i$.

The same holds for $y' = \frac{y}{y+1}$.

Zilber's Principle for DCF

Theorem (Hrushovski-Sokolović)

If $X \subseteq \mathbb{K}^n$ is strongly minimal and non-locally modular, then $X \not\perp C$.

The original proof used the high powered model theoretic machinery of *Zariski Geometries* developed by Hrushovski and Zilber.

This was later given a more elementary conceptual proof by Pillay and Ziegler.

Nontrivial Modular Strongly Minimal Sets in DCF

Where do we look for nontrivial modular strongly minimal sets?

- By Hrushovski's result we should look for a modular strongly minimal group G.
- By results of Pillay's we may assume that $G \subseteq H$ where H is an algebraic group and the Zariski closure or G is H.
- By strong minimality we may assume that H is commutative and has no proper infinite algebraic subgroups.
- H is either $\mathbb{G}_a, \mathbb{G}_m$ or a simple abelian variety.
- If H is $\mathbb{G}_a$, $\mathbb{G}_m$ or isomorphic to an abelian variety defined over the constants, then $H \not\perp C$.

Modular Strongly Minimal Sets in DCF

Theorem (Hrushovski-Sokolović)

- *If A is a simple abelian variety that is not isomorphic to one defined over C and $A^\sharp$ is the Kolchin-closure of the torsion points, then $A^\sharp$ is a modular strongly minimal set. (called a* Manin kernel*)*
- *If X is a modular strongly minimal set, then there is A as above such that $X \not\perp A^\sharp$.*
- *$A_0^\sharp \not\perp A_1^\sharp$ if and only if A_0 and A_1 are isogenous.*

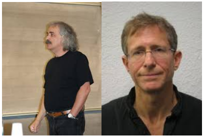
The key tool is the Buium–Manin homomorphism, a differential algebraic homomorphism $\mu : A \to \mathbb{K}^n$ such that $\ker(\mu) = A^\sharp$ and the result that $A^\sharp$ is Zariski dense and has no proper proper infinite differential algebraic subgroups.

Application: The number of countable DCF

In 1984 Shelah proved Vaught's Conjecture for ω-stable theories, but it took almost ten years to show $I(\mathrm{DCF}, \aleph_0) = 2^{\aleph_0}$.

We can assign a dimension to $A^\sharp$ which can be finite or infinite. As these dimensions can be assigned independently we can code graphs into DCF. (eni-DOP)

Diophantine Applications



The strongly minimal sets $A^\sharp$ play a fundamental role in Buium's and Hrushovski's proofs of the Mordell-Lang Conjecture for function fields in characteristic 0.

Corollary

If A is a simple abelian variety not isomorphic to a variety defined over C with $\dim(A) \geq 2$ and $X \subset A$ is a curve, then X contains only finitely many torsion points.

Proof If $X \cap A^\sharp$ is infinite and $A^\sharp$ is strongly minimal, $X \cap A^\sharp$ is cofinite in $A^\sharp$ and hence Zariski dense in A, a contradiction.

Rigid DCF

Theorem (M)

There are countable differentially closed fields with no non-trivial automorphisms.

Trivial Pursuits?

In DCF we have a very good understanding of the non-locally modluar ($\not\perp C$) and non-trivial locally modular ($\not\perp$ Manin kernel) strongly minimal sets.

What can we say about trivial strongly minimal sets?

Trivial Pursuits

So far there is no good theory of the trivial strongly minimal sets.

Look for examples:

- Rosenlicht style examples: $y' = f(y)$, f a rational function over C. We can determine triviality by studying the partial fraction decomposition of $1/f$.
- Hrushovski–Itai 2004: For X a curve of genus at least 2 defined over C there is a trivial $Y \subset X$ such that $\mathbb{K}(X) = \mathbb{K}\langle Y \rangle$.

  They use these ideas to build many different ω-stable theories of differential fields.

Painlevé Equations

Painlevé began the classification of second order differential equations where the only movable singularities are poles. The classification gives rise six families:

For example $P_{II}(\alpha) : D^{(2)}Y = 2Y^3 + tY + \alpha$ where $D(t) = 1$.

Many questions arise about algebraic relations between solutions to an individual equation and relations between solutions of different equations. These were attacked by Nishioka, Umemura, Wantanabe...

Generic Painlevé Equations

Nagloo and Pillay 2014 gave a model theoretic interpretation of this earlier work and used model theoretic ideas to significantly extend it. For example,

Theorem

If $\alpha \in \mathbb{C}$ is transcendental, then the solution set of $P_{II}(\alpha)$ is strongly minimal trivial and if $y_1, \ldots, y_n$ are distinct solutions, $y_1, \ldots, y_n$ are algebraically independent.

The j-function

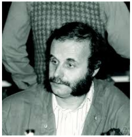
Theorem (Freitag–Scanlon)

The is a third order non-linear differential equation $E(y)$ satisfied by the j-function is strongly minimal, trivial but not $\aleph_0$–categorical.

The proof relies on Pila's Ax–Lindemann–Wierstrass Theorem for the j-function with derivates.

This has been greatly extended by recent work of Blázquez-Sanz, Casale, Freitag, Nagloo been on the differential equations satisfied by uniformizing functions for Fuchsian groups

Poizat's Example



Let $V = \{x : x' = xx''\}$

Theorem (Poizat)

The only infinite irreducible differential algebraic subvariety of V is given by $X' = 0$.

Corollary

$\frac{X''}{X'} = \frac{1}{X}$ *defines a trivial strongly minimal set.*

Fact: Any strongly minimal set defined over C of order at least 2 is trivial.
order 2 $\Rightarrow \perp C$;
defined over $C \Rightarrow \perp A^\sharp$ for A a Manin kernel;

Generalized Poizat Equations

Joint work with Jim Freitag, Rémi Jaoui and Ronnie Nagloo.
For $f(z) \in \mathbb{C}(z)$ consider the differential equation $\frac{z''}{z'} = f(z)$.
Let V_f denote the solutions.

Theorem

V_f is strongly minimal if and only there is no $g \in \mathbb{C}(z)$ with $f = \frac{dg}{dz}$.

$\Rightarrow$ If $f = \frac{dg}{dz}$ and $z' = g(z) + c$ for some $c \in C$, then $z'' = f(z)z'$.
Thus there is an infinite family of order 1 differential algebraic subvarieties of V_f.

Suppose $f(z) \in \mathbb{C}(z)$ has no antiderivative in $\mathbb{C}(z)$.

Partial Fractions Any $f(z) \in \mathbb{C}(z)$ can be expressed

$$f(z) = g'(z) + \sum_{i=1}^{n} \frac{c_i}{z - \alpha_i}$$

where $g(z) \in \mathbb{C}(z)$, and $c_1, \ldots, c_n, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

f has an antiderivative in $\mathbb{C}(z)$ if and only if $n = 0$.

Suppose $f(z)$ has no an antiderivative.
By a change of variables, we may assume some $\alpha_i = 0$.
Consider the power series expansion

$$f(z) = \sum_{n=m}^{\infty} a_n z^m$$

then $a_{-1} \neq 0$. We call a_{-1} the *residue* at 0.

If V_f is not strongly minimal, then we can find a differential field (K, δ) and $z \in V_f$ transcendental over K such that z and z' are algebraically dependent over K.

Consider the Puiseux series field $K\langle\langle z\rangle\rangle$.
Since this field is algebraically closed we can identify z' as some series u in $K\langle\langle z\rangle\rangle$

Define a derivation D on $K\langle\langle z\rangle\rangle$ such that

$$D\left(\sum a_i z^i\right) = \sum \delta(a_i) z^i + u \sum i a_i z^{i-1}$$

D extends the natural derivation on $K(z, z')$.

Let $u = \sum_{i=0}^{\infty} a_i z^{r+\frac{i}{n}}$, where $v(u) = r$.
Then

$$z'' = D(u) = \sum \delta(a_i) z^{r+\frac{i}{n}} + u \sum \left(r + \frac{i}{n} \right) a_i z^{r+\frac{i}{n}-1}$$

Since

$$v\left(\sum \delta(a_i) z^{r+\frac{i}{n}} \right) \geq r,$$

$$\frac{z''}{z'} = \frac{D(u)}{u} = \alpha + \sum \left(r + \frac{i}{n} \right) a_i z^{r+\frac{i}{n}-1}$$

where $v(\alpha) \geq 0$.

The coefficient of z^{-1} on the right summand is 0.
Thus we can not have $\frac{z''}{z'} = f(z)$, a contradiction.

Liénard equations

Liénard equations are of the form

$$z'' + f(z)z' + g(z) = 0$$

where $f, g \in \mathbb{C}(z)$ arise in oscillating circuits and have applications in mechanics, seismology, chemistry and cosmology.

Corollary

Suppose f has no antiderivative in $\mathbb{C}(z)$. Consider

$$z'' + f(z)z' + \frac{\sum_{i=0}^{n} c_i z^i}{\sum_{j=0}^{m} d_j z^j} = 0$$

where $c_0, \dots, c_n, d_0, \dots, d_m$ are algebraically independent constants. The generic type is orthogonal to the constants.

Consider

$$z'' + f(z)z' + \frac{\sum_{i=0}^{n} c_i z^i}{\sum_{j=0}^{m} d_j z^j} = 0$$

A specialization theorem of Jaoui says that for certain families of definable sets over C, if the generic fiber is non-orthogonal to the constants, then all fibers are non-orthogonal to C.

Here $z'' + f(z)z' = 0$ is orthogonal to the constants, so the generic fiber is as well.

Algebraic relations between solutions

Suppose $f, g \in \mathbb{C}(z)$ have no antiderivatives in $\mathbb{C}$ (possibly $f = g$). Suppose $K \supseteq \mathbb{C}$ is a differential field $x \in V_f$, $y \in V_g$ are each transcendental over K.

Theorem

If $K(x, x')^{\mathrm{alg}} = K(y, y')^{\mathrm{alg}}$, *then* $\mathbb{C}(x)^{\mathrm{alg}} = \mathbb{C}(y)^{\mathrm{alg}}$.

In particular, if $y \in cl(x)$, then $y \in \mathbb{C}(x)^{\mathrm{alg}}$.

Ingredients:

- basics on trivial strongly minimal sets;
- connections between Kähler differentials and transcendence, building on the work of Ax, Rosenlicht and Brestovski.

Suppose $x \in V_f$, $y \in V_g$ and y is algebraic over $\mathbb{C}(x)$.
$x'' = f(x)x'$ and $y'' = g(y)y'$.
There is an algebraic function ϕ, such that $\phi(x) = y$.

$$\begin{aligned} y' &= \phi'(x)x' \\ y'' &= \phi''(x)(x')^2 + \phi'(x)x'' \\ y'g(y) &= \phi''(x)(x')^2 + \phi'(x)x'f(x) \\ \phi'(x)x'(g(\phi(x)) &= \phi''(x)(x')^2 + \phi'(x)x'f(x) \\ \phi''(x)x' &= \phi'(x)(f(x) - g(\phi(x))) \end{aligned}$$

As x satisfies no non-trivial first order differential equation we must have $\phi''(x) = 0$ and thus $\phi(x) = ax + b$.
Since $\phi'(x) = a \neq 0$.
We must also have $f(x) = g(\phi(x))$.

Lemma

i) If $f \neq g$ *and* $V_f \not\perp V_g$*, then* $g = f \circ \phi$ *for some affine transformation* $\phi(x) = ax + b$*;*
ii) If $x, y \in V_f$ *and* $y \in \mathrm{cl}(x)$*, there is* ϕ *as above with* $f = f \circ \phi$ *and* $\phi(x) = y$*.*

For most f there in no nontrivial ϕ with $f = f \circ \phi$, so $\mathrm{cl}(x) = \{x\}$.
But it is possible

1) $f(x) = \frac{1}{x-a} - \frac{1}{x-b}$ and $\phi(x) = -x + a + b$;
2) $f(x) = \frac{1}{(x-1)(x-\eta)\cdots(x-\eta^{n-1})}$ where $\eta^n = 1$ and $\phi(x) = \eta x$

Suppose $f(x)$ has nonzero residues at $\alpha_1, \ldots, \alpha_n$.
If $f = f \circ \phi$ then ϕ permutes $\alpha_1, \ldots, \alpha_n$.

The group of transformations $x \mapsto ax + b$ acts sharply 2-transitively on $\mathbb{C}$.
Thus there are at most $n(n-1)$ ϕ permuting $\alpha_1, \ldots, \alpha_n$.

Corollary

In V_f, $|\mathrm{cl}(x)| \leq n(n-1)$. Thus V_f is $\aleph_0$-categorical.

This can be sharpened from $n(n-1)$ to n with more detailed analysis.

The non strongly minimal case

Suppose $\frac{dg}{dz} = f$. Then for each constant c, $z' = g(z) + c$ is a differential subvariety of V_f. So V_f has rank 2.

There are three possibilities.
1) V_f is internal to the constants; example $f(z) = a$;

2) Each $z' = g(z) + c$ is non-orthogonal to the constants, but V_f is 2-step analyzable but not internal to the constants; example $f(z) = az + b$;

3) Generic fibers $z' = g(z) + c$ are orthogonal to the constants; example if $f(z) = z^2 + az + b$ where a and b are algebraically independent over $\mathbb{Q}$, then generic fibers are orthogonal and orthogonal to $\mathbb{C}$.

Thank You