MATH 215 **Written Homework 5 Solution** **07/21/2009**

1. (**20 points total**) This is very similar to WH4, Problem 1.

(a) When $n = 1$ the equation holds since both sides are $A \times A_1$ in this case. (**1**)

Suppose $n \geq 1$ and the equation holds for all sets $A, A_1, \ldots, A_n$ and let $A, A_1, \ldots, A_{n+1}$ be sets. The equation holds for $n = 2$ (this we are allowed to assume). Thus we may assume $n \geq 2$. (**1**) Using the fact that the equation holds for $n = 2$ and the induction hypothesis we calculate

$$\begin{aligned} A \times (A_1 \cup \cdots \cup A_{n+1}) &= A \times ((A_1 \cup \cdots \cup A_n) \cup A_{n+1}) \quad (\mathbf{1}) \\ &= (A \times (A_1 \cup \cdots \cup A_n)) \cup (A \times A_{n+1}) \quad (\mathbf{1}) \\ &= ((A \times A_1) \cup \cdots \cup (A \times A_n)) \cup (A \times A_{n+1}) \quad (\mathbf{1}) \\ &= (A \times A_1) \cup \cdots \cup (A \times A_{n+1}) \quad (\mathbf{1}) \end{aligned}$$

which means that the equation holds for $A, A_1, \ldots, A_{n+1}$. (**1**) By induction the equation holds for all sets $A, A_1, \ldots, A_n$, where $n \geq 1$. (**1**)

(b) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (**4**) and then repeat the proof of part (a) with "$\cap$" replacing "$\cup$"; graded same way. (**8**)

2. (**20 points total**) This requires a bit of patience since there are so many cases. It should have been stated that A and B are not empty.

Each of the statements implies itself (**2**). We consider the other implications. For counter examples we take $A = B = \mathbf{R}$. (**3** for the correct number of cases.) We repeat the statements for convenience and refer to statements by their labels.

(a) $\forall a \in A,\ \exists b \in B,\ P(a, b)$;

(b) $\exists b \in B,\ \forall a \in A,\ P(a, b)$;

(c) $\exists a \in A,\ \forall b \in B,$ not $P(a, b)$;

(d) $\exists a \in A,\ \exists b \in B,\ P(a, b)$.

(a) $\not\Longrightarrow$ (b). (**3**) Take $P(a, b) : a \geq b$ for example. Then (a) is true ($\forall a \in A$ take $b = a - 1$) but (b) is false since $\forall b \in B$ the statement $a \geq b$ is false with $a = b - 1$.

(a) $\not\Longrightarrow$ (c). (**3**) The statements of (a) and (c) are negations of each other.

(a) $\Longrightarrow$ (d). (**3**) Note $\forall a \in A, Q(a)$ implies $\exists a \in A, Q(a)$ holds for non-empty sets A.

(b) $\Longrightarrow$ (a). (**2**) Observe (b) can be read for some $b_0 \in B$ the statement $P(a, b_0)$ is true for all $a \in A$. Thus for all $a \in A$ there is a $b \in B$, namely $b = b_0$, such that $P(a, b)$ is true. Note in (a) the $b \in B$ mentioned may very well depend on the $a \in A$.

(b) $\not\Longrightarrow$ (c). (**2**) Take $P(a,b) : a^2 \geq 0$ for example, which is always true. Thus "not $P(a,b)$" is always false.

(b) $\Longrightarrow$ (d). (**2**) Note (b) implies "$\exists b \in B,\ \exists a \in A,\ P(a,b)$", since A is not empty, and the latter is equivalent to (d).

(c) $\not\Longrightarrow$ (a). Take $P(a,b) : a^2 < 0$, for example, which is false. Thus "not $P(a,b)$" is true.

(c) $\not\Longrightarrow$ (b). Same.

(c) $\not\Longrightarrow$ (d). Same.

(d) $\not\Longrightarrow$ (a). Take $P(a,b) : a = 0$.

(d) $\not\Longrightarrow$ (b). Take $P(a,b) : a \geq b$.

(d) $\not\Longrightarrow$ (c). Take $P(a,b) : a^2 \geq 0$.

3. (**20 points total**) For $x \neq 4$ observe that $|f(x) - 41| = |(11x - 3) - 41| = |11x - 44| = 11|x - 4|$. (**3**) Let $\epsilon > 0$ (**3**) and $\delta = \epsilon/11$ (**3**). Then

$$\begin{aligned} 0 < |x - 4| < \delta &\Longrightarrow 0 < |x - 4| < \epsilon/11 \ \ (\mathbf{3}) \\ &\Longrightarrow 0 < 11|x - 4| < \epsilon \ \ (\mathbf{3}) \\ &\Longrightarrow 0 < |f(x) - 41| < \epsilon \ \ (\mathbf{3}) \\ &\Longrightarrow |f(x) - 41| < \epsilon \ \ (\mathbf{2}). \end{aligned}$$

4. (**20 points total**) $\lim_{x \longrightarrow a} f(x) = b$ is the statement
"$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbf{R},\ 0 < |x - a| < \delta$ implies $|f(x) - b| < \epsilon$".

(a) The negation of the statement is:
"$\exists \epsilon > 0$ (**2**), $\forall \delta > 0$ (**2**), $\exists x \in \mathbf{R}$ (**2**), $0 < |x - a| < \delta$ (**2**) and (**2**) $|f(x) - b| \geq \epsilon$ (**2**)".

(b) Here is an argument. Let $\delta > 0$ and $x = \pm\delta/2$. Then $0 < |x - 0| = \delta/2 < \delta$. Note $|f(-\delta/2) - b| = |1/3 - b|$ and $|f(\delta/2) - b| = |1/2 - b|$. One of $|1/3 - b|, |1/2 - b|$ is positive, else $1/3 = b = 1/2$, a contradiction. Let ϵ be the smallest positive value of the previous line. Then $|f(-\delta/2) - b| = \epsilon \geq \epsilon$ or $|f(\delta/2) - b| = \epsilon \geq \epsilon$. Thus the statement of (a) is satisfied with $x = -\delta/2$ or $x = \delta/2$. (**8**)

5. (**20 points total**) $f : \mathbf{R} \longrightarrow \mathbf{R}$ is given by $f(x) = x^2 - 6x + 21$.

(a) Completing the square we see $f(x) = (x - 3)^2 + 12 \geq 12$. Therefore $f(x) \neq 11.99$ for all $x \in \mathbf{R}$, for example. (**10**) We have shown that f is not surjective.

Comment: Need a specific $y \in \mathbf{R}$ such that $f(x) \neq y$ for all $x \in \mathbf{R}$.

(b) $f(x) = x(x - 6) + 21$ so $f(0) = 21 = f(6)$. (**10**) Therefore f is not injective.

Comment: Need specific $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.