MATH 180 **Quiz 8 (Version I) 03/16/06** **Solutions 03/20/06**

Name (print) ____________________________ **Discussion hour (T Th** ______**)**

1. (10 pts.) Let $f(x) = x^3 + ax^2 + bx + c$.

- Find the parameters a, b and c, given that the y-intercept is 1, and that f has local extrema at $x = -4$ and $x = 2$.

  Solution : Given that the y-intercept is 1, we immediately get that $c = 1$. Moreover, the derivative function is $f'(x) = 3x^2 + 2ax + b$ which is defined for every x in $\mathbb{R}$. Therefore, the existence of local extrema at the given positions $x = -4$ and $x = 2$, implies corresponding zeros for the derivative f'. We have the following:

$$\begin{cases} f'(-4) = 0 \\ f'(2) = 0 \end{cases} \Rightarrow \begin{cases} 3(-4)^2 - 8a + b = 0 \\ 3(2)^2 + 4a + b = 0 \end{cases} \Rightarrow \begin{cases} 8a - b = 48 \\ 4a + b = -12 \end{cases} \Rightarrow \begin{cases} a = 3 \\ b = -24 \end{cases}$$

  Note: We can solve the 2×2 linear system with any known method (i.e. substitution or elimination) in order to find the parameters a, b.
  So, the answer is $\boxed{(a, b, c) = (3, -24, 1)}$ which gives the following formula for the function: $\boxed{f(x) = x^3 + 3x^2 - 24x + 1}$. (**5 pts**.)

- Identify the local extrema of f as minima or maxima.

  Solution : Having found the parameters, we get that $f'(x) = 3x^2 + 6x - 24$. Moreover, the second derivative is $f''(x) = 6x + 6$. So, we can use the f''- test as follows:

$$\begin{cases} f''(-4) = -18 < 0 \\ f''(2) = 18 > 0 \end{cases} \Rightarrow \begin{cases} \text{At } x = -4,\ f \text{ has a local maximum.} \\ \text{At } x = 2\ ,\ f \text{ has a local minimum.} \end{cases} \quad (\textbf{5 pts.})$$

  Note: We could have used the f'- test i.e. check whether f' changes sign around the two given x positions. It doesn't matter which method you used, as long as you got the correct answer.

2. (10 pts.) Let $f(x) = \ln(1 + x^2)$, for $-1 \le x \le 2$. Find the best possible bounds for the function.

Solution : We are interested in the closed interval $[-1, 2]$. First, we find the local extrema of the function.

$$f'(x) = \frac{1}{1 + x^2} \cdot \left(1 + x^2\right)' \quad \Rightarrow \quad f'(x) = \frac{2x}{1 + x^2}$$

The derivative function is defined for all real numbers (note that the denominator is always positive) and so, it is certainly defined for all x in $[-1,2]$. Therefore, in order to find the critical points, we have to solve the equation $f'(x)=0$:

$$f'(x)=0 \quad \Rightarrow \quad \frac{2x}{1+x^2}=0 \quad \Rightarrow \quad 2x=0 \quad \Rightarrow \quad x=0.$$

So, for the given interval, the function has only one critical point which occurs at the position $x=0$. Using the first derivative test, we get that:

- For $x<0$, i.e. for x in $[-1,0)$, $f'(x)<0$.

- For $x>0$, i.e. for x in $(0,2]$, $f'(x)>0$.

and therefore, at $x=0$ the function has a **local minimum**, which is the value $f(0)=0$.
Now, it remains to check the values that the function gets at the endpoints of the given interval. We have that $f(-1)=\ln 2$ and $f(2)=\ln 5$.
So, for the given interval, we conclude that f has at $x=2$ a **global maximum**, which is the value $f(2)=\ln 5$, and has at $x=0$ a **global minimum**, which is the value $f(0)=0$.
Therefore, the answer is (in terms of bounds) : $\boxed{0 \leq y \leq \ln 5}$, for all x in $[-1,2]$. (**10 pts.**)

*** END OF VERSION I ***
*** OVER FOR VERSION 2 ***

MATH 180 **Quiz 8 (Version 2) 03/16/06** **Solutions 03/20/06**

Name (print) ____________________________ **Discussion hour (T Th** ______**)**

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1. (10 pts.) Let $f(x) = x^3 + ax^2 + bx + c$.

- Find the parameters a, b and c, given that the y-intercept is 2, and that f has local extrema at $x = -3$ and $x = 1$.

  Solution : Given that the y-intercept is 2, we immediately get that $c = 2$. Moreover, the derivative function is $f'(x) = 3x^2 + 2ax + b$ which is defined for every x in $\mathbb{R}$. Therefore, the existence of local extrema at the given positions $x = -3$ and $x = 1$, implies corresponding zeros for the derivative f'. We have the following:

$$\begin{cases} f'(-3) = 0 \\ f'(1) = 0 \end{cases} \Rightarrow \begin{cases} 3(-3)^2 - 6a + b = 0 \\ 3(1)^2 + 2a + b = 0 \end{cases} \Rightarrow \begin{cases} 6a - b = 27 \\ 2a + b = -3 \end{cases} \Rightarrow \begin{cases} a = 3 \\ b = -9 \end{cases}$$

  Note: We can solve the 2×2 linear system with any known method (i.e. substitution or elimination) in order to find the parameters a, b.
  So, the answer is $\boxed{(a, b, c) = (3, -9, 2)}$ which gives the following formula for the function: $\boxed{f(x) = x^3 + 3x^2 - 9x + 2}$. (**5 pts.**)

- Identify the local extrema of f as minima or maxima.

  Solution : Having found the parameters, we get that $f'(x) = 3x^2 + 6x - 9$. Moreover, the second derivative is $f''(x) = 6x + 6$. So, we can use the f''- test as follows:

$$\begin{cases} f''(-3) = -12 < 0 \\ f''(1) = 12 > 0 \end{cases} \Rightarrow \begin{cases} \text{At } x = -3, \ f \text{ has a local maximum.} \\ \text{At } x = 1 \ , \ f \text{ has a local minimum.} \end{cases} \quad (\textbf{5 pts.})$$

  Note: We could have used the f'- test i.e. check whether f' changes sign around the two given x positions. It doesn't matter which method you used, as long as you got the correct answer.

2. (10 pts.) Let $f(x) = \ln(1 + 2x^2)$, for $-1 \le x \le 2$. Find the best possible bounds for the function.

Solution : We are interested in the closed interval $[-1, 2]$. First, we find the local extrema of the function.

$$f'(x) = \frac{1}{1 + 2x^2} \cdot \left(1 + 2x^2\right)' \quad \Rightarrow \quad f'(x) = \frac{4x}{1 + 2x^2}$$

The derivative function is defined for all real numbers (note that the denominator is always positive) and so, it is certainly defined for all x in $[-1, 2]$. Therefore, in order to find the critical points, we have to solve the equation $f'(x) = 0$:

$$f'(x) = 0 \quad \Rightarrow \quad \frac{4x}{1+2x^2} = 0 \quad \Rightarrow \quad 4x = 0 \quad \Rightarrow \quad x = 0.$$

So, for the given interval, the function has only one critical point which occurs at the position $x = 0$. Using the first derivative test, we get that:

- For $x < 0$, i.e. for x in $[-1, 0)$, $f'(x) < 0$.

- For $x > 0$, i.e. for x in $(0, 2]$, $f'(x) > 0$.

and therefore, at $x = 0$ the function has a **local minimum**, which is the value $f(0) = 0$. Now, it remains to check the values that the function gets at the endpoints of the given interval. We have that $f(-1) = \ln 3$ and $f(2) = \ln 9$.
So, for the given interval, we conclude that f has at $x = 2$ a **global maximum**, which is the value $f(2) = \ln 9$, and has at $x = 0$ a **global minimum**, which is the value $f(0) = 0$. Therefore, the answer is (in terms of bounds) : $\boxed{0 \leq y \leq \ln 9}$, for all x in $[-1, 2]$. (**10 pts.**)

*** END OF VERSION 2 ***