

# The parameterized family of polynomials $f(x) = x^4 + ax^2$

Let  $a$  be a fixed real number. We will analyze the graph of the fourth degree polynomial  $y = f(x) = x^4 + ax^2$  by considering two cases. First we note that

$$f(x) = x^4 + ax^2 = x^2(x^2 + a), \quad (1)$$

$$f'(x) = 4x^3 + 2ax = 2x(2x^2 + a), \quad (2)$$

$$f''(x) = 12x^2 + 2a = 2(6x^2 + a). \quad (3)$$

There are two natural cases to consider.

*Case 1:*  $a \geq 0$ . Using (1) we note that  $f(x) > 0$  when  $x \neq 0$  and  $f(x) = 0$  when  $x = 0$ .

$$\frac{f \quad \text{positive}}{f \quad + \quad 0 \quad +} \text{positive}.$$

Since  $2x^2 + a \geq x^2 \geq 0$  for all  $x$  we conclude from (2) that  $f'(x) = 0$  if and only if  $x = 0$  and

$$\frac{f \quad \text{decreasing}}{f' \quad - \quad 0 \quad +} \text{increasing}.$$

Since  $6x^2 + a \geq 0$  for all  $x$  we conclude from (3) that  $f''(x) \geq 0$  for all  $x$  and  $f''(x) = 0$  if and only if  $x = 0$  and  $a = 0$ . Thus if  $a > 0$  we have

$$\frac{(\text{graph of}) f \quad \text{concave up}}{f'' \quad +}$$

and if  $a = 0$  we have

$$\frac{(\text{graph of}) f \quad \text{concave up}}{f'' \quad + \quad 0 \quad +} \text{concave up}.$$

In any event  $f'$  is increasing on  $(-\infty, 0]$  and  $[0, \infty)$  since its derivative  $f''$  is positive on  $(-\infty, 0)$  and  $(0, \infty)$ . Therefore  $f'$  is increasing on  $(-\infty, \infty)$  which is to say that  $f'$  is always increasing. Our conclusion: the graph of  $f$  is *always* concave up.

Use your graphing calculator to plot the graphs of  $y = f(x)$  where  $a = 0, 0.5, 1, 3$  on the same screen so that they can be compared.

*Case 2:  $a < 0$ .* By (1) we note that  $f(x) = 0$  if and only if  $x^2 = 0$  or  $x^2 + a = 0$ ; that is  $x = 0, -\sqrt{-a}$ , or  $\sqrt{-a}$ . Since the graph of the quadratic  $y = x^2 + a$  opens upward and crosses the  $x$ -axis at  $-\sqrt{-a}, \sqrt{-a}$ , we see from (1) that

$$\begin{array}{ccccccc} f & \text{positive} & & \text{negative} & & \text{negative} & \text{positive} \\ \hline f & + & -\sqrt{-a} & - & 0 & - & \sqrt{-a} & + \end{array}$$

By (2) we see that  $f'(x) = 0$  if and only if  $2x = 0$  or  $2x^2 + a = 0$ ; that is  $x = 0, -\sqrt{\frac{-a}{2}}$ , or  $\sqrt{\frac{-a}{2}}$ . Since the graph of the quadratic  $y = 2x^2 + a$  opens upward and crosses the  $x$ -axis at  $x = -\sqrt{\frac{-a}{2}}, \sqrt{\frac{-a}{2}}$ , by virtue of (2)

$$\begin{array}{ccccccc} f & \text{decreasing} & & \text{increasing} & & \text{decreasing} & \text{increasing} \\ \hline f' & - & -\sqrt{\frac{-a}{2}} & + & 0 & - & \sqrt{\frac{-a}{2}} & + \end{array}$$

In particular  $f(x)$  has local minima which occur at  $x = -\sqrt{\frac{-a}{2}}, \sqrt{\frac{-a}{2}}$  and  $f(x)$  has one local maximum which occurs at  $x = 0$ . Since

$$f\left(-\sqrt{\frac{-a}{2}}\right) = f\left(\sqrt{\frac{-a}{2}}\right) = \left(\sqrt{\frac{-a}{2}}\right)^2\left(\left(\sqrt{\frac{-a}{2}}\right)^2 + a\right) = \left(\frac{-a}{2}\right)\left(\frac{-a}{2} + a\right) = -\frac{a^2}{4}$$

and  $f(0) = 0$ ,

$$\boxed{\text{local minima: } \left(-\sqrt{\frac{-a}{2}}, -\frac{a^2}{4}\right), \left(\sqrt{\frac{-a}{2}}, -\frac{a^2}{4}\right)}$$

$$\boxed{\text{local maxima: } (0, 0).}$$

Observe that  $f(x)$  has no maximum, but

$$\boxed{f(x) \text{ has a minimum which is } -\frac{a^2}{4}.}$$

By (3) we see that  $f''(x) = 0$  if and only if  $6x^2 + a = 0$  or equivalently  $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$ . Since the graph of the quadratic  $y = 6x^2 + a$  opens upward and crosses the  $x$ -axis at  $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$ , as a consequence of (3)

(graph of) $f$	concave up		concave down		concave up
$f''$	+	$-\sqrt{\frac{-a}{6}}$	-	$\sqrt{\frac{-a}{6}}$	+

Therefore the graph of  $y = f(x)$  has two inflection points which are at  $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$ . Since

$$f\left(-\sqrt{\frac{-a}{6}}\right) = f\left(\sqrt{\frac{-a}{6}}\right) = \left(\sqrt{\frac{-a}{6}}\right)^2 \left(\left(\sqrt{\frac{-a}{6}}\right)^2 + a\right) = \left(\frac{-a}{6}\right) \left(\frac{-a}{6} + a\right) = -\frac{5a}{36},$$

inflection points:  $\left(-\sqrt{\frac{-a}{6}}, -\frac{5a}{36}\right), \left(\sqrt{\frac{-a}{6}}, -\frac{5a}{36}\right)$

Use your graphing calculator to plot the graphs of  $y = f(x)$  where  $a = -0.5, -1, -8$  on the same screen so that they can be compared.