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Representations of Certain Classes of Hopf Algebras

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0. Introduction

Let k be an algebraically closed field of characteristic 0. Recent classification results for certain large classes of pointed Hopf algebras by Andruskiewitsch and Schneider show that generalizations of quantized enveloping algebras and the small quantum groups of Lusztig cover quite a bit of ground.

We discuss generalizations of complete reducibility theorems for the quantized enveloping algebras. The arguments follow those in Lusztig's book to a good extent. We describe aspects of nearly completed work by Andruskiewitsch, Radford, and Schneider.

As background we will discuss the ideas behind the classification results. We will also describe a basic class of modules which is at the heart of the theory of simple modules for our Hopf algebras of interest. This is based on joint work of Radford and Schneider.

1. The Andruskiewitsch–Schneider Classification Program

This is a program to classify Hopf algebras whose coradical H_0 is a sub-Hopf algebra.

Step 1: Pass from H to the coradically graded Hopf algebra $\mathrm{gr}\,(H)$.

Step 2: Determine the possible Hopf algebra structures for $\mathrm{gr}\,(H)$.

Since $\mathrm{gr}\,(H)_0 = H_0$, Step 2 is to treat the graded case. In passing from H to $\mathrm{gr}\,(H)$ information may be lost. $\mathrm{gr}\,(H) = R\#H_0$ is a bi-product ($\mathrm{gr}\,(H) \overset{\longleftarrow}{\longrightarrow} H_0$).

Step 3: **Lifting**: Given a particular Hopf algebra structure for $\mathrm{gr}\,(H)$, determine the possible Hopf algebra structures for H.

Two illustrations, where $k=\mathbb{C}$. Let $n,m>1$, where $n|m$, and $q,\alpha\in k$, where $q=\sqrt[n]{1}$.

$$\Delta(a)=a\otimes a \quad\text{and}\quad \Delta(z)=z\otimes 1+a\otimes z,$$

where $z=x,y$.

Example 1. *Partial algebra structure:*

In H*:* $ax=qxa$, $\boxed{x^n=\alpha(a^n-1)}$, $a^m=1$;

In gr(H)*:* $\mathbf{ax}=q\mathbf{xa}$, $\boxed{\mathbf{x}^n=\mathbf{0}}$, $\mathbf{a}^m=\mathbf{1}$;

where $\mathbf{a}=a$, $\mathbf{1}=1$, $\mathbf{x}=x+H_0$.

Example 2. *Partial algebra structure:*

In H*:* $ax=qxa$, $ay=q^{-1}ya$, $\boxed{xy-q^{-1}yx=a^2-1}$;

In gr(H)*:* $\mathbf{ax}=q\mathbf{xa}$, $\mathbf{ay}=q^{-1}\mathbf{ya}$, $\boxed{\mathbf{xy}-q^{-1}\mathbf{yx}=\mathbf{0}}$;

Remark 1. $xy-q^{-1}yx$ *is a commutator in a certain context.*

From this point on $k=\overline{k}$ *and* char $k=0$.

2. R

From now on: H is pointed, $\Gamma = G(H)$ is abelian, $\mathrm{gr}\,(H) \xrightarrow{\pi} H_0 = k[\Gamma]$ is the projection.

$$\boxed{R = \{h \in \mathrm{gr}\,(H) \,|\, h_{(1)} \otimes \pi(h_{(2)}) = h \otimes 1\}.}$$

R **is a subalgebra of** H, $\Delta(R) \subseteq \mathrm{gr}\,(H) \otimes R$,

$$R \otimes k[\Gamma] \longrightarrow \mathrm{gr}\,(H) \ \ (r \otimes h \mapsto rh)$$

is a linear isomorphism. $\boxed{\pi(rh) = \epsilon(r)h.}$ Define $\Pi : \mathrm{gr}\,(H) \longrightarrow R$ by $\boxed{\Pi(rh) = r\epsilon(h)}$.

$\boxed{\Pi^2 = \Pi, \ \mathrm{Im}\,\Pi = R}$, and R **is a coalgebra**

$$\Delta_R = (\Pi \otimes \Pi)(\Delta | R), \varepsilon = \epsilon | R.$$

$\Pi : \mathrm{gr}\,(H) \longrightarrow R$ is an onto coalgebra map.

- R is **not** always a Hopf algebra over k; the coproduct Δ_R is can fail to be an algebra map.

R is a left $k[\Gamma]$-module/comodule where

$$h{\cdot}r = h_{(1)} r S(h_{(2)}),$$

$$\rho(r) = r_{(-1)} \otimes r_{(0)} = \pi(r_{(1)}) \otimes r_{(2)}.$$

$$\boxed{\rho(h{\cdot}r) = h_{(1)} r_{(-1)} S(h_{(3)}) \otimes h_{(2)}{\cdot}r_{(0)}} \qquad (1)$$

for $h \in k[\Gamma]$, $r \in R$ holds. Braided monoidal category: ${}^{\Gamma}_{\Gamma}\mathcal{Y}D = {}^{k[\Gamma]}_{k[\Gamma]}\mathcal{Y}D$: *objects* left $k[\Gamma]$-modules and comodules such that (1) holds. *Braiding* (iso)morphisms: $M \otimes N \xrightarrow{\sigma_{M,N}} N \otimes M$ given by

$$\boxed{\sigma_{M,N}(m \otimes n) = m_{(-1)}{\cdot}n \otimes m_{(0)}}.$$

$A, B \in {}^{\Gamma}_{\Gamma}\mathcal{Y}D$ algebras. $A \otimes B$ a k-algebra with

$$\boxed{m = (m_A \otimes m_B)(\mathrm{Id} \otimes \sigma_{B,A} \otimes \mathrm{Id})},$$

$$(a \otimes \underline{b}) \otimes (\underline{a'} \otimes b') \mapsto aa' \otimes bb',\ 1_k \mapsto 1_A \otimes 1_B.$$

Write $A \underline{\otimes} B$ for this structure, $a \underline{\otimes} b = a \otimes b$.

$$\boxed{(a \underline{\otimes} b)(a' \underline{\otimes} b') = a(b_{(-1)}{\cdot}a') \underline{\otimes} b_{(0)} b'.}$$

• **The coproduct** $R \xrightarrow{\Delta_R} R\underline{\otimes}R$ **is an algebra morphism and** R **is a Hopf algebra of** ${}^{\Gamma}_{\Gamma}\mathcal{Y}D$.

The **bi-product** $R\#k[\Gamma] \overset{v.s.}{=} R\otimes k[\Gamma]$ is defined, the smash product and coproduct, and is a **Hopf algebra over** k. The linear isomorphism

$$R\#k[\Gamma] \longrightarrow \mathrm{gr}\,(H) \ \ (r\#h = r\otimes h \mapsto rh)$$

is an isomorphism of Hopf algebras.

The appropriate notion of "commutator" in ${}^{\Gamma}_{\Gamma}\mathcal{Y}D$ is **braided commutator**. For $a, b \in R$ the usual commutator

$$\mathrm{ad}\, a\ (b) = [a, b] = ab - ba = m_R(\mathrm{Id} - \tau_{R,R})(a\otimes b).$$

Replacing $\tau_{R,R}$ by $\sigma_{R,R}$

$$\boxed{\mathrm{ad}_c\, a\ (b) = [a, b]_c = ab - (a_{(-1)}\cdot b)a_{(0)}}$$

and $\mathrm{ad}_c\, a$ is a morphism.

3. Nichols Algebras

For $n \geq 0$ set $\mathrm{gr}\,(H)(n) = H_n/H_{n-1}$ and

$$\boxed{R(n) = R \cap \mathrm{gr}\,(H)(n).}$$

Then $R = \bigoplus_{n=0}^{\infty} R(n)$ is a coradically graded Hopf algebra of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

(N.1) $R = \bigoplus_{n=0}^{\infty} R(n)$ is a graded pointed irreducible Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$;

(N.2) $P(R) = R(1)$; and possibly

(N.3) $R(1)$ generates R as an algebra.

An algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ satisfying (N.1)–(N.3) is a **Nichols algebra**. [**Nichols 78**, **A-S 02**]

- The **subalgebra of** R generated by $V = R(1)$ is a **Nichols algebra** $\mathcal{B}(V)$.
- **Determining whether or not** $R = \mathcal{B}(V)$ **is a major problem in the classification program**.

If $W \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ there exists $\mathcal{B}(W) \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with $W = \mathcal{B}(W)(1)$.

Let $\Gamma = G(H)$. Then $V = \bigoplus_{g\in\Gamma} V_g$, where

$$\boxed{V_g = \{v \in V \mid \rho(v) = g\otimes v\}.}$$

Suppose V is finite-dimensional and Γ acts on V as diagonalizable operators, e.g. H f-dim.

The module/comodule condition for $h \in \Gamma$ and $v \in V_g$ is $\boxed{\rho(h\cdot v) = hgh^{-1}\otimes h\cdot v = g\otimes h\cdot v}$. V_g is a Γ-submodule; has weight space decomposition. There exists a basis $\{x_1, \ldots, x_n\}$ for V, and $g_1, \ldots, g_n \in \Gamma$, characters $\chi_1, \ldots, \chi_n \in \widehat{\Gamma}$, such that for $1 \le i \le n$ and $g \in \Gamma$

$$\boxed{\rho(x_i) = g_i\otimes x_i \ \text{ and } \ g\cdot x_i = \chi_i(g)x_i.} \tag{2}$$

$$\mathbf{c := \sigma_{V,V} \ \text{ and } \ q_{ij} := \chi_j(g_i).}$$

$$c(\underline{x}_i\otimes x_j) = \underline{g}_i\cdot x_j\otimes\underline{x}_i = \chi_j(g_i)x_j\otimes x_i = q_{ij}x_j\otimes x_i;$$

$$\boxed{c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.}$$

(q_{ij}) := **infinitesimal braiding matrix**.

$c : V \otimes V \longrightarrow V \otimes V$:= **infinitesimal braiding**.

$$\boxed{\mathrm{ad}_c\, x_i\, (x_j) = [x_i, x_j]_c = x_i x_j - q_{ij} x_j x_i.}$$

Remark 2. Γ *an abelian group,* $g_1, \dots, g_n \in \Gamma$ *and* $\chi_1, \dots, \chi_n \in \widehat{\Gamma}$*.* W *with basis* $\{x_1, \dots, x_n\}$ *such that (2) is satisfied for* $1 \leq i \leq n$ *and* $g \in \Gamma$ *is an object of* ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ *and thus determines the Nichols algebra* $\mathcal{B}(W)$*.*

A braiding matrix (q_{ij}) is of **Cartan type** if

$$\boxed{q_{ij} q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \neq 1}$$

for all i, j where a_{ij} are integers with $a_{ii} = 2$ and $-\mathrm{ord}(q_{ii}) < a_{ij} \leq 0$ for all $i \neq j$. [**A-S 00**]

Example 3. *From* $\mathrm{U}_q(\mathfrak{g})$*,* $\mathfrak{g}$ *a semisimple Lie algebra with Cartan matrix* (a_{ij})*,* $q_{ij} = q^{d_i a_{ij}}$*.*

.

4. Classification Results

Theorem 1 (6.2, A-S 07). *H f.-d. pointed, $\Gamma = G(H)$ abelian, $\forall\ i$ $\mathrm{ord}(q_{ii}) > \mathbf{7}$ is odd, $q_{i\ell}q_{\ell i} = q_{ii}^{-3}, q_{\ell\ell}^{-3}$ for some ℓ implies $3 \nmid \mathrm{ord}(q_{ii})$. Then $H \simeq u(\mathcal{D}, \lambda, \mu)$ for some $\mathcal{D}, \lambda, \mu$.* $\square$

$i, j \in I = \{1, \ldots, n\}$. **Parameters of** $u(\mathcal{D}, \lambda, \mu)$:

Datum of finite Cartan type $\mathcal{D}(\Gamma, \{g_i\}, \{\chi_i\}, (a_{ij}))$.
$q_{ij} = \chi_j(g_i)$, $\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}, \chi_i(g_i) \neq 1$.

I := {points of Dynkin diagram}, $\sim$ on I defines components, $\mathbb{X}$:= {components}.

Linking parameters $\lambda = \{\lambda_{ij}\} \subseteq k$. $\lambda_{ji} = -q_{ij}^{-1}\lambda_{ij}$; $\lambda_{ij} \neq 0$ implies $i \not\sim j$, $g_i \neq g_j^{-1}$, and $\chi_i = \chi_j^{-1}$.

Root vector parameters $\mu = \{\mu_\alpha\}_{\alpha \in \Phi^+} \subseteq k$.

V with basis $\{x_1, \ldots, x_n\}$. Then $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ where $g{\cdot}x_i = \chi_i(g)x_i$ and $\rho(x_i) = g_i \otimes x_i$ for all $g \in \Gamma$, $i \in I$. $T(V)$ is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

<u>$U(\mathcal{D}, \lambda) := T(V)\#k[\Gamma]$ mod the relations</u>

$$
\boxed{
\begin{aligned}
(QSR)\ \ & (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, i \sim j, i \neq j; \\
(LR)\ \ & (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = [x_i\, x_j]_c \\
& \qquad = x_i x_j - q_{ij} x_j x_i = \lambda_{ij}(1 - g_i g_j), i \not\sim j.
\end{aligned}
}
$$

$gx_ig^{-1} = \chi_i(g)x_i, \Delta(g) = g \otimes g, \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$

for $g \in \Gamma$, $i \in I$.

<u>$u(\mathcal{D}, \lambda, \mu) := U(\mathcal{D}, \lambda)$ mod the relations</u>

$$
\boxed{(RVR)\ \ x_\alpha^{N_J} = u_\alpha(\mu) \in k[G(H)], \alpha \in \Phi_J^+, J \in \mathbb{X}.}
$$

$U(\mathcal{D},\lambda)$'s account for $U_q(\mathfrak{g})$'s, $\mathfrak{g}$ a semisimple Lie algebra, in the **generic case**; $\chi_i(g_i) = q_{ii}$ is not a root of unity for $i \in I$.

$\mathcal{D} = (\Gamma, \{g_i\}, \{\chi_i\}, (a_{i\,j}))$ is a **datum of finite Cartan type**. [**A-S 07**]

• If ord$(q_{i\,i})$ is prime to $2, 3$ then • gr$(u(\mathcal{D},\lambda,\mu)) \simeq \mathcal{B}(V)\#k[\Gamma]$. [**A-S 07**]

$U(\mathcal{D},\lambda)$ is a quotient of a **generalized double** when the **linking graph is bipartite**, for example • in the generic case, or • when the Cartan matrix is simply laced and ord$(q_{i\,i})$ is odd for $i \in I$, or • for all $i \in I$ there is a $j \in I$ with $\lambda_{ij} \neq 0$ and ord$(q_{i\,i}) > 3$. [**Radford-Schneider 07**]

The linking graph $\mathbf{G}(\lambda,\mathcal{D})$: points = components $\mathbf{x}$ of the Dynkin diagram of (a_{ij}); edge $\mathbf{x}-\mathbf{y}$ if $\lambda_{ij}\neq 0$ for some $i\in\mathbf{x}$, $j\in\mathbf{y}$.

Assume that $\lambda\neq 0$ and $\mathbf{G}(\lambda,\mathcal{D})$ is bipartite. Let $\mathbb{X}=\mathbb{X}^{-}\cup\mathbb{X}^{+}$ be a bipartite decomposition. Let Λ be the free abelian group on $\{z_i\}_{i\in I}$, let

$$I^{-}=\bigcup_{\mathbf{x}\in\mathbb{X}^{-}}\mathbf{x}\quad\text{and}\quad I^{+}=\bigcup_{\mathbf{x}\in\mathbb{X}^{+}}\mathbf{x};$$

- $\eta_i\in\widehat{\Lambda}$ $\qquad\eta_i(z_j)=\chi_i(g_j),\ \ i,j\in I^{-}$;
- $\mathcal{D}^{-}=\mathcal{D}(\Lambda,\{z_i\}_{i\in I^{-}},\{\eta_i\}_{i\in I^{-}},(a_{ij})_{i,j\in I^{-}})$
- $\mathcal{D}^{+}=\mathcal{D}(\Gamma,\{g_i\}_{i\in I^{+}},\{\chi_i\}_{i\in I^{+}},(a_{ij})_{i,j\in I^{+}})$,
- $U=\mathcal{B}(V^{-})\#k[\Lambda]$ and $A=\mathcal{B}(V^{+})\#k[\Gamma]$.

Restrictions $\operatorname{Alg}(A,k)\longrightarrow\widehat{\Gamma}$, $\operatorname{Alg}(U,k)\longrightarrow\widehat{\Lambda}$ are bijective. For $\chi\in\widehat{\Gamma}$ let $L(\chi)=L(\chi,\rho)\in{}_{U(\mathcal{D},\lambda)}\mathcal{M}$ be as defined in §5, where

$$\rho(z_i)=\chi(g_i)\qquad\text{for } i\in I^{-}.$$

The character χ is **dominant for** $(\mathcal{D}, \lambda, I^+)$ if for all there are integers $m_i \geq 0$ for $i \in I^-$ such that $j \in I^+$ and $\lambda_{ij} \neq 0$ implies $\chi(g_i g_j) = q_{ij}^{m_i}$. [**R-S 07**]

Theorem 2 (R-S 07). *In the generic case*

$$U(\mathcal{D}, \lambda) \simeq (U \otimes A)_\sigma / (z_i \otimes g_i - 1 \otimes 1 \,|\, i \in I^-),$$

where σ is a 2-cocycle for $U \otimes A$. $\chi \mapsto [L(\chi)]$ determines a bijection between the dominant characters and the isomorphism classes of finite-dimensional left $U(\mathcal{D}, \lambda)$-modules. □

There is a similar result for $u(\mathcal{D}, \lambda, 0)$; **here** $x_\alpha^{N_J} = 0$ **for all** $\alpha \in \Phi_J^+$, $J \in \mathbb{X}$. "the characters" replaces "dominant characters".

The linking graph of $u(\mathcal{D}, \lambda, 0)$ may not be bipartite; however $u(\mathcal{D}, \lambda, 0)$ can be replaced with $u(\mathcal{D}', \lambda', 0)$ which has the same finite-dimensional irreducibles and whose linking graph is bipartite.

5. Generalized Doubles

U, A are Hopf algebras and $U \otimes A \xrightarrow{\tau} k$ satisfies

$$\tau_\ell : U \longrightarrow A^* \qquad \text{and} \qquad \tau_r : A \longrightarrow U^{*\,op}$$

are algebra maps, where

$$\tau_\ell(u)(a) = \tau(u, a) = \tau_r(a)(u)$$

for $u \in U$, $a \in A$. Then $(U \otimes A) \otimes (U \otimes A) \xrightarrow{\sigma} k$,

$$(u \otimes a) \otimes (u' \otimes a') \mapsto \epsilon(u)\tau(u', a)\epsilon(a')$$

is a 2-cocycle and thus determines a Hopf algebra $(U \otimes A)^\sigma = U \otimes A$ as a coalgebra with multiplication given by

$$\begin{aligned} &(u \otimes a)(u' \otimes a') \\ &\quad = \; u\tau(u'_{(1)}, a_{(1)})u'_{(2)} \otimes a_{(2)}\tau^{-1}(u'_{(3)}, a_{(3)})a' \end{aligned}$$

for all $u, u' \in U$ and $a, a' \in A$. This product satisfies

$$\boxed{(u \otimes a)(u' \otimes a') = uu' \otimes aa' \text{ when } a = 1 \text{ or } u' = 1.}$$

Example 4 (Doi-Takeuchi 94). *H f.-d., $U = (H^{op})^*$, $A = H$, and $\tau(p, a) = p(a)$ for all $p \in U$ and $a \in A$. Then $(U \otimes A)^\sigma = D(H)$ is the quantum double of H.*

Hopf algebras of the type $(U \otimes A)_\sigma$ are **generalized doubles**. [**Doi-Takeuchi 94**, **Joseph 95**] The representation theory of §6 applies to generalized doubles.

6. An Abstract Highest Weight Theory

[**R-S 07**] $\mathbb{H}$ is an algebra with subalgebras U, A

$$U \otimes A \xrightarrow{\mu} \mathbb{H} \quad (u \otimes a \mapsto ua) \qquad (3)$$

is bijective. Commutation rule:

$$\boxed{au = \sum_{i=1}^{n} u_i a_i \quad \text{where} \quad \mu^{-1}(au) = \sum_{i=1}^{n} u_i \otimes a_i.}$$

With the identification of (3) the vector space $U \otimes A$ has an algebra structure such that

$$\boxed{(u \otimes a)(u' \otimes a') = uu' \otimes aa' \text{ when } a = 1 \text{ or } u' = 1.}$$

Let $\rho \in \mathrm{Alg}\,(U, k)$. Then $(A, \triangleleft_\rho) \in \mathcal{M}_{\mathbb{H}}$, where

$$\boxed{a \triangleleft_\rho ua' = \Big((\rho \otimes \mathrm{Id})(\mu^{-1}(au)\Big)\, a' = \Big(\sum_{i=1}^{n} \rho(u_i) a_i\Big)\, a'}$$

for all $a, a' \in A$ and $u \in U$. If $au = ua$ observe that $a \triangleright_\rho ua' = \rho(u) aa'$. The module structure results from regarding $k \in \mathcal{M}_U$ by $1 \cdot u = \rho(u)1$ and identifying A with $k \otimes_U \mathbb{H}$ by

$$k \otimes_U \mathbb{H} \simeq k \otimes_U (U \otimes A) \simeq (k \otimes_U U) \otimes A \simeq k \otimes A \simeq A.$$

Let $\chi \in \mathsf{Alg}\,(A,k)$. Then $(U, \triangleright_\chi) \in {}_{\mathbb{H}}\mathcal{M}$, where

$$\boxed{ua \triangleright_\chi u' = u\Big((\mathrm{Id}{\otimes}\chi)(\mu^{-1}(au'))\Big) = u\left(\sum_{i=1}^n u'_i\chi(a_i)\right)}$$

for all $u, u' \in U$ and $a \in A$. If $au' = u'a$ then $ua \triangleright_\chi u' = \chi(a)uu'$. In particular

$$\boxed{u \triangleright_\chi u' = uu' \quad \text{and} \quad a \triangleright_\chi 1 = \chi(a)1.}$$

Let $I(\rho, \chi)$ be the largest $\mathbb{H}$-submodule of U contained in $\mathsf{Ker}\,\rho$ and set

$$L(\rho, \chi) = U/I(\rho, \chi) \in {}_{\mathbb{H}}\mathcal{M}.$$

Then $L(\rho, \chi)$ has a codim. one U-submodule N, where U acts on $L(\rho, \chi)/N$ by ρ, and $L(\rho, \chi)$ is a cyclic $\mathbb{H}$-module generated by a dim. one A-submodule km, where A acts on km by χ. These assertions follow since the same are true for $(U, \triangleright_\chi)$ with $N = \mathsf{Ker}\,\rho$ and $m = 1$.

Reversing the roles of χ and ρ we obtain $J(\chi, \rho)$ and $R(\chi, \rho) = A/J(\chi, \rho) \in \mathcal{M}_{\mathbb{H}}$ of a similar description.

Define a bilinear form $A \times U \xrightarrow{\Psi} k$ by

$$\Psi(a,u) = (\rho \otimes \chi)(\mu^{-1}(au)) = \sum_{i=1}^{n} \rho(u_i)\chi(a_i)$$

for all $a \in A$ and $u \in U$.

Lemma 1. *The following hold for the form:*

(a) $\Psi(a \triangleleft_\rho h, u) = \Psi(a, h \triangleright_\chi u)$ *for all* $a \in A$, $h \in \mathbb{H}$, *and* $u \in U$; *that is* Ψ *is* $\mathbb{H}$*-balanced.*

(b) $A^\perp = \{u \in U \,|\, \Psi(A,u) = (0)\} = I(\rho,\chi)$.

(c) $U^\perp = \{a \in A \,|\, \Psi(a,U) = (0)\} = J(\chi,\rho)$.

$\square$

Denote the form $A/U^\perp \times U/A^\perp \longrightarrow k$ by Ψ also. Since $R(\rho,\chi) \times L(\chi,\rho) \xrightarrow{\Psi} k$ is non-singular and $\mathbb{H}$-balanced we may regard $R(\rho,\chi)$ as a submodule of $L(\chi,\rho)^*$ and $L(\chi,\rho)$ as a submodule of $R(\rho,\chi)^*$. This is useful for computation.

Lemma 2. *Suppose there are subalgebras U' of U and A' of A such that:*

(a) $a \triangleright_\chi u' = \chi(a)u'$ *for all* $a \in A, u' \in U'$ *and* $\mathsf{Alg}\,(U,k) \to \mathsf{Alg}\,(U',k)$ $(\eta \mapsto \eta|U')$ *is injective.*

(b) $a' \triangleleft_\rho u = \rho(u)a'$ *for all* $a' \in A', u \in U$ *and* $\mathsf{Alg}\,(A,k) \to \mathsf{Alg}\,(A',k)$ $(\eta \mapsto \eta|A')$ *is injective.*

Then $L(\rho,\chi)$ contains a unique dim. one A-submodule and a unique codim. one U-submodule and $R(\chi,\rho)$ contains a unique dim. one U-submodule and a unique codim. one A-submodule. □

Let ${}_{\mathbb{H}}\underline{\mathcal{M}}$ be the full subcategory of ${}_{\mathbb{H}}\mathcal{M}$ whose modules M are

(1) generated by a dim. one left A-submodule km and

(2) have a codim. one left U-submodule which contains no non-zero $\mathbb{H}$-submodules.

$L(\chi, \rho) \in {}_{\mathbb{H}}\underline{\mathcal{M}}$ and every object is isomorphic to an $L(\chi, \rho)$.

Theorem 3. *Assume the hypothesis of the preceding lemma. Then the map*

$$\mathsf{Alg}\,(U, k) \times \mathsf{Alg}\,(A, k) \to [{}_{H}\underline{\mathcal{M}}], \quad (\rho, \chi) \mapsto [L(\rho, \chi)]$$

is bijective. $\square$

Corollary 1. *Assume the hypothesis of the preceding lemma and also that the finite dimensional simple left U-modules and A-modules have dimension one. Then the finite-dimensional $L(\rho, \chi)$'s and the finite-dimensional simple $\mathbb{H}$-modules are one in the same.* $\square$

Reversing roles of U and A we can define $\underline{\mathcal{M}}_{\mathbb{H}}$ and develop a duality between ${}_{\mathbb{H}}\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}_{\mathbb{H}}$.

7. Perfect Linkings and Reduced Data

A linking parameter λ of a datum $\mathcal{D}$ is *perfect* if any vertex is linked.

Any linking parameter λ the Hopf algebra $\mathcal{U}(\mathcal{D},\lambda)$ has a natural quotient Hopf algebra $\mathcal{U}(\mathcal{D}',\lambda')$ with perfect linking parameter λ'. This is the special case of

A *reduced datum* is

$$\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}),$$

where Γ is an abelian group, θ is a positive integer, $K_i, L_i \in \Gamma, \chi_i \in \widehat{\Gamma}$ for all $1 \le i \le \theta$ satisfy

$$\chi_j(K_i) = \chi_i(L_j) \text{ for all } 1 \le i,j \le \theta, \tag{4}$$

$$K_iL_i \neq 1 \text{ for all } 1 \le i \le \theta. \tag{5}$$

A reduced datum $\mathcal{D}_{red}$ is called *generic* if for all $1 \le i \le \theta$, $\chi_i(K_i)$ is not a root of unity.

A *linking parameter* l for a reduced datum $\mathcal{D}_{red}$ is a family $l = (l_i)_{1 \le i \le \theta}$ of non-zero elements in $\Bbbk$.

Lemma 3. *Let*

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \le i \le \theta}, (\chi_i)_{1 \le i \le \theta})$$

be a datum satisfying $\chi_j(g_i)\chi_i(g_j) \neq \chi_i(g_i)^2$ *for all* $i \neq j$, *and let* λ *be a perfect linking parameter for* $\mathcal{D}$. *Then there are a reduced datum* $\mathcal{D}_{red}$ *and a linking parameter* l *for* $\mathcal{D}_{red}$ *such that*

$$\mathcal{U}(\mathcal{D}, \lambda) \cong \mathcal{U}(\mathcal{D}_{red}, l)$$

as Hopf algebras.

Set

$$\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, l).$$

Let ad_l and ad_r be the adjoint actions of the Hopf algebra $k\langle u_1, \ldots, a_\theta \rangle \# k[\Gamma]$. Then $U(\mathcal{D}_{red}, l)$

is generated by Γ and $E_1, \ldots, E_\theta, F_1, \ldots, F_\theta$ subject to the relations for Γ and

$$\begin{aligned}
&\mathrm{ad}_l(E_i)^{1-a_{ij}}(E_j) = 0, \ \forall\, 1 \leq i, j \leq \theta, i \neq j,\\
&\mathrm{ad}_r(F_i)^{1-a_{ij}}(F_j) = 0, \ \forall\, 1 \leq i, j \leq \theta, i \neq j,\\
&\quad E_iF_j - F_jE_i = \delta_{ij} l_i (K_i - L_i^{-1}), \ \ \forall\, 1 \leq i, j \leq \theta.
\end{aligned}$$

Explicitly $\forall \ i \in J, J \in \mathcal{X}$ and $1 \leq j \leq \theta, i \neq j$,

$$\begin{aligned}
&\mathrm{ad}_l(E_i)^{1-a_{ij}}(E_j)\\
&= \sum_{s=0}^{1-a_{ij}} (-p_{ij})^s \left[1 - a_{ij}\right]_{q_J^{d_i}} E_i^{1-a_{ij}-s} E_j E_i^s,\\
&\mathrm{ad}_r(F_i)^{1-a_{ij}}(F_j)\\
&= \sum_{s=0}^{1-a_{ij}} (-p_{ij})^s \left[1 - a_{ij}\right]_{q_J^{d_i}} F_i^s F_j F_i^{1-a_{ij}-s}.
\end{aligned}$$

The action of Γ is given for all $g \in \Gamma$ and all $1 \leq i \leq \theta$ by

$$\begin{aligned}
gE_ig^{-1} &= \chi_i(g)E_i,\\
gF_ig^{-1} &= \chi_i^{-1}(g)F_i,
\end{aligned}$$

and the comultiplication by

$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1,$$
$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1}.$$

$\mathbf{U}^+$ is generated by the E_i's and $\mathbf{U}^-$ by the F_i's; Γ the group generated by the K_i's, L_i's.

Corollary 2. *The multiplication map*

$$\mathbf{U}^- \otimes \mathbf{U}^+ \otimes k[\Gamma] \to \mathbf{U}$$

is an isomorphism of vector spaces. Furthermore $\mathbf{U}^- = \mathfrak{B}(W)$ *and* $\mathbf{U}^+ = \mathfrak{B}(V)$ *for some* $V, W \in {}_{\Gamma}^{\Gamma}\mathcal{YD}$.

Set $\mathbb{I} = \{1, \dots, \theta\}$. Let $\mathbb{Z}[\mathbb{I}]$ be a free abelian group of rank θ with fixed basis $\alpha_1, \dots, \alpha_\theta$, and $\mathbb{N}[\mathbb{I}] = \{\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \mid n_1, \dots, n_\theta \in \mathbb{N}\}$.

For $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}[\mathbb{I}], n_1, \dots, n_\theta \in \mathbb{Z}$, let $|\alpha| = \sum_{i=1}^{\theta} n_i$, and

$\chi_\alpha = \chi_1^{n_1} \cdots \chi_\theta^{n_\theta}, K_\alpha = K_1^{n_1} \cdots K_\theta^{n_\theta}, L_\alpha = L_1^{n_1} \cdots L_\theta^{n_\theta}.$

There is a $\mathbb{k}$-bilinear form $(\ ,\ ) : \mathbf{U}^- \otimes \mathbf{U}^+ \to \mathbb{k}$, non-degenerate on $\mathbf{U}^-_{-\alpha} \times \mathbf{U}^+_{\alpha}$.

For all $\alpha \in \mathbb{N}[\mathbb{I}]$, let $x^k_\alpha, 1 \leq k \leq d_\alpha = \dim \mathbf{U}^-_{-\alpha}$, be a basis of $\mathbf{U}^-_{-\alpha}$, and $y^k_\alpha, 1 \leq k \leq d_\alpha$, the dual basis of $\mathbf{U}^+_\alpha$ with respect to $(\ ,\ )$. Set

$$\theta_\alpha = \sum_{k=1}^{d_\alpha} x^k_\alpha \otimes y^k_\alpha.$$

We will set $\theta_\alpha = 0$ for all $\alpha \in \mathbb{Z}[\mathbb{I}]$ and $\alpha \notin \mathbb{N}[\mathbb{I}]$. $\Omega = \sum_{\alpha \in \mathbb{N}[\mathbb{I}]} \sum_{k=1}^{d_\alpha} S(x^k_\alpha) y^k_\alpha = \sum_{\alpha \in \mathbb{N}[\mathbb{I}]} \theta_\alpha$ is the *quasi-R-matrix*.

Theorem 4. *Let $\alpha \in \mathbb{N}[\mathbb{I}]$ and $1 \leq i \leq \theta$. Then in $\mathbf{U} \otimes \mathbf{U}$,*

$$\begin{aligned}
&(E_i \otimes 1)\theta_\alpha + (K_i \otimes E_i)\theta_{\alpha-\alpha_i} \\
&\quad = \ \theta_\alpha(E_i \otimes 1) + \theta_{\alpha-\alpha_i}(L_i^{-1} \otimes E_i), \\
&(1 \otimes F_i)\theta_\alpha + (F_i \otimes L_i^{-1})\theta_{\alpha-\alpha_i} \\
&\quad = \ \theta_\alpha(1 \otimes F_i) + \theta_{\alpha-\alpha_i}(F_i \otimes K_i).
\end{aligned}$$

8. Classes of $\mathbf{U}$-modules

$\mathcal{D}_{red}$ is generic, regular, and of Cartan type, and $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, l)$. Regular means: Let Q be the subgroup of $\widehat{\Gamma}$ generated by $\chi_1, \ldots, \chi_\theta$. Then

$$\mathbb{Z}[\mathbb{I}] \xrightarrow{\cong} Q,\ \alpha \mapsto \chi_\alpha,$$

is an isomorphism.

$\mathcal{C}$ is the full subcategory of ${}_{\mathbf{U}}\mathcal{M}$ of all left $\mathbf{U}$-modules M which are direct sums of 1-dim. Γ-modules; i.e. have a weight space decomposition $M = \oplus_{\chi \in \widehat{\Gamma}} M^\chi$,

$$M^\chi = \{m \in M \mid gm = \chi(g)m \text{ for all } g \in \Gamma\}$$

Weight for M: a character $\chi \in \widehat{\Gamma}$ with $M^\chi \neq 0$.

$\mathcal{C}^{hi}$ the full subcategory of $\mathcal{C}$; $M \in \mathcal{C}^{hi}$ if for any $m \in M$, $\exists\ N \geq 0$ such that $\mathbf{U}^+_\alpha m = 0$ for all $\alpha \in \mathbb{N}[\mathbb{I}]$ with $|\alpha| \geq N$.

Both $\mathcal{C}$ and $\mathcal{C}^{hi}$ are closed under sub-objects and quotient objects of $\mathbf{U}$-modules.

$\chi \in \widehat{\Gamma}$. The *Verma module*

$$M(\chi) = \mathbf{U}/(\sum_{i=1}^{\theta} \mathbf{U}E_i + \sum_{g \in \Gamma} \mathbf{U}(g - \chi(g)).$$

The inclusion $\mathbf{U}^- \subset \mathbf{U}$ defines a $\mathbf{U}^-$-module isomorphism

$$\mathbf{U}^- \xrightarrow{\cong} M(\chi) = \mathbf{U}/(\sum_{i=1}^{\theta} \mathbf{U}E_i + \sum_{g \in \Gamma} \mathbf{U}(g - \chi(g))) \tag{6}$$

by the triangular decomposition of $\mathbf{U}$.

The Verma module $M(\chi)$ and all its quotients belong to the category $\mathcal{C}^{hi}$.

Regularity allows a partial order $\leq$ on $\widehat{\Gamma}$. For all $\chi, \chi' \in \widehat{\Gamma}$, $\chi' \leq \chi$ if $\chi = \chi'\chi_\alpha$ for some $\alpha \in \mathbb{N}[\mathbb{I}]$. If $\chi' \leq \chi$ then χ and χ' are in the same Q-coset of $\widehat{\Gamma}$.

Lemma 4. *Let $\chi \in \widehat{\Gamma}$, and $M \in \mathcal{C}$. Suppose χ is a maximal weight for M and $m \in M^{\chi}$. Then $E_i m = 0$ for all $1 \le i \le \theta$, and $\mathbf{U}m$ is a quotient of $M(\chi)$.*

Let $M \in \mathcal{C}$ and C be a coset of Q in $\widehat{\Gamma}$. Then $M_C = \oplus_{\chi \in C} M^{\chi} \in \mathcal{C}$. Note $M = \oplus_C M_C$, where C runs over the Q-cosets of $\widehat{\Gamma}$.

Lemma 5. *Suppose $M \in \mathcal{C}^{hi}$ is a fin.-gen. $\mathbf{U}$-module.*

1. $\operatorname{Dim} M^{\chi} < \infty$ *for all $\chi \in \widehat{\Gamma}$.*
2. *For all $\chi' \in \widehat{\Gamma}$ there are only finitely many weights χ for M with $\chi' \le \chi$.*
3. *Every non-empty set of weights for M has a maximal element.*

$M \in \mathcal{C}$ is *integrable* for all $m \in M$ and $1 \le i \le \theta$, $E_i^n m = F_i^n m = 0$ for some $n \ge 1$.

$\chi \in \widehat{\Gamma}$ is *dominant* if for all $1 \leq i \leq \theta$ there are $m_i \geq 0$ such that $\chi(K_i L_i) = q_{ii}^{m_i}$ for all $1 \leq i \leq \theta$. $\widehat{\Gamma}^+ =$ dominant characters of $\widehat{\Gamma}$.

Let $\chi \in \widehat{\Gamma}^+$, and $m_i \geq 0$ for all $1 \leq i \leq \theta$ such that $\chi(K_i L_i) = q_{ii}^{m_i}$ for all $1 \leq i \leq \theta$. Set

$$L(\chi) = \mathbf{U}/(\sum_{i=1}^{\theta} \mathbf{U}E_i + \sum_{i=1}^{\theta} \mathbf{U}F_i^{m_i+1} + \sum_{g \in \Gamma} \mathbf{U}(g - \chi(g))).$$

A universal property of the Verma module with respect to integrable modules in $\mathcal{C}$:

$$\boxed{\text{Proposition 1.}}$$

Proposition 1. *Let $M \in \mathcal{C}$ be integrable and $\chi \in \widehat{\Gamma}$. Assume that there exists an element $0 \neq m \in M^\chi$ such that $E_i m = 0$ for all $1 \leq i \leq \theta$. Then χ is dominant, and there is a unique $\mathbf{U}$-linear map $t : L(\chi) \to M$ such that $t(l_\chi) = m$.*

Corollary 3. *Under the assumptions above:*

1. *Let χ be dominant. Then*

$$\mathbf{U}^-/(\sum_{i=1}^{\theta} \mathbf{U}^- F_i^{m_i+1}) \simeq L(\chi),$$

   $L(\chi)$ is integrable, and $L(\chi)^\chi$ is a one-dimensional vector space with basis l_χ.

2. *Let $\chi, \chi' \in \widehat{\Gamma}^+$. Then $L(\chi) \simeq L(\chi')$ iff $\chi = \chi'$.*

9. The quantum Casimir Operator

Again, $\mathcal{D}_{red}$ is generic, regular, and of Cartan type, and $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, l)$. In addition we *assume that the family $(q_{ii})_{1 \le i \le \theta}$ of scalars in $\Bbbk$ is $\mathbb{N}$-linearly independent* with respect to multiplication.

The q_{ii}'s are $\mathbb{N}$-linearly independent if $\mathbb{I}$ is connected, that is if the Cartan matrix of $\mathcal{D}_{red}$ is indecomposable.

Lemma 6. *Let C be a coset of Q in $\widehat{\Gamma}$.*

1. *There is a function $G : C \to k^{\times}$ such that for all $\chi \in C$ and $1 \le i \le \theta$, $G(\chi) = G(\chi\chi_i^{-1})\chi(K_iL_i)$. G is uniquely determined up to multiplication by a non-zero constant in $\Bbbk$.*
2. *Let G be as in (1). If $\chi, \chi' \in \widehat{\Gamma}^+$ are dominant characters with $\chi \ge \chi'$ and $G(\chi) = G(\chi')$, then $\chi = \chi'$.*

Proposition 2. *Let C be a coset of Q in $\widehat{\Gamma}$, and $M = M_C \in \mathcal{C}^{hi}$. Choose a function G as in Lemma 6 and define a $\Bbbk$-linear map $\Omega_G : M \to M$ by $\Omega_G(m) = G(\chi)\Omega(m)$ for all $m \in M^\chi, \chi \in C$.*

1. *Ω_G is $\mathbf{U}$-linear and locally finite.*
2. *Suppose $0 \neq m \in M$ generates a quotient of a Verma module $M(\chi)$, $\chi \in \widehat{\Gamma}$. Then $\chi \in C$, and $\Omega_G(m) = G(\chi)m$.*
3. *The eigenvalues of Ω_G are the $G(\chi)'s$, where χ runs over the maximal weights of the submodules N of M (in which case $\Omega_G(n) = G(\chi)n$ for all $n \in N^\chi$).*

The function $\Omega_G : M \to M$ in Proposition 2 is called the *quantum Casimir operator*.

11. Complete Reducibility Theorems

$\mathcal{D}_{red}$ is generic, regular, and of Cartan type, and $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, l)$.

Theorem 5. *Let $\chi \in \widehat{\Gamma}^+$.*

1. *$L(\chi)$ is a simple $\mathbf{U}$-module.*
2. *Any weight vector of $L(\chi)$ which is annihilated by all $E_i, 1 \leq i \leq \theta$, is a scalar multiple of l_χ.*

Theorem 6. *Let M be an integrable module in $\mathcal{C}^{hi}$. Then M is completely reducible and M is a direct sum of $L(\chi)'s$ where $\chi \in \widehat{\Gamma}^+$.*

Corollary 4. $\mathbf{U}$ *is Γ-reductive.*

Recall that Γ^2 denotes the subgroup of Γ generated by the products $K_1 L_1, \ldots, K_\theta L_\theta$.

Theorem 7. *Assume that $\mathcal{D}_{red}$ is regular. Then the following are equivalent:*

1. $\mathbf{U}$ *is reductive.*
2. $[\Gamma : \Gamma^2]$ *is finite.*

If $\mathbf{U}$ is reductive, then the Cartan matrix of $\mathcal{D}_{red}$ is invertible.

11. Reductive Pointed Hopf Algebras

$B \subseteq A$ a subalgebra. A is B-*reductive* if every fin.-dim. left A-module which is B-semisimple (on restriction) is A-semisimple. A pointed, $\Gamma = G(A)$; Γ-reductive means $\Bbbk[\Gamma]$-reductive.

Theorem 8. *Let $\mathcal{D}$ be a generic datum of finite Cartan type.*

1. *The following are equivalent:*
   (a) *$U(\mathcal{D}, \lambda)$ is Γ-reductive.*
   (b) *The linking parameter λ of $\mathcal{D}$ is perfect.*
2. *The following are equivalent:*
   (a) *$U(\mathcal{D}, \lambda)$ is reductive.*
   (b) *The linking parameter λ of $\mathcal{D}$ is perfect and $[\Gamma : \Gamma^2]$ is finite.*