MATH 180 Hour Exam I Review Solutions Radford 02/11/02

1. $f(x) = (3x^2 + 4x - 1)/(4x + 3)$

   a) This is a calculator exercise. The values of x listed indicate that calculation of (two sided) limits involves approach form *both* sides. Observe that no approximation is the actual limit.

   b) $\lim_{x\to 1}(4x+3) = \lim_{x\to 1}(4x) + \lim_{x\to 1} 3 = (\lim_{x\to 1} 4)(\lim_{x\to 1} x) + \lim_{x\to 1} 3 = 4(1)+3 = 7$; thus $\lim_{x\to 1}(4x+3) = 7$. Since this limit is not zero,

$$
\begin{aligned}
\lim_{x\to 1}(3x^2 + 4x - 1)/(4x + 3) &= \lim_{x\to 1}(3x^2 + 4x - 1)/\lim_{x\to 1}(4x + 3) \\
&= \lim_{x\to 1}(3x^2 + 4x - 1)/3 \\
&= [\lim_{x\to 1}(3x^2) + \lim_{x\to 1}(4x) + \lim_{x\to 1}(-1)]/3 \\
&= [\lim_{x\to 1}(3)(\lim_{x\to 1} x^2) + \lim_{x\to 1}(4)(\lim_{x\to 1} x) + \lim_{x\to 1}(-1)]/3 \\
&= [\lim_{x\to 1}(3)(\lim_{x\to 1} x)^2 + \lim_{x\to 1}(4)(\lim_{x\to 1} x) + \lim_{x\to 1}(-1)]/3 \\
&= [(3)(1)^2 + 4(1) + (-1)]/3 = 6/7.
\end{aligned}
$$

   c) Under the continuity assumption, $\lim_{x\longrightarrow 1} f(x) = f(1) = 6/7$.

2. $f(x) = \begin{cases} 3x^2 + 2x^3 + 7 & : \ x < 0 \\ 6x^4 + x^3 + 2 & : \ x \geq 0 \end{cases}$

   a) Polynomials are continuous. Thus $\lim_{x\longrightarrow 0} 3x^2 + 2x^3 + 7 = 3(0)^2 + 2(0)^3 + 7 = 7$.

   b) Ditto. Thus $\lim_{x\longrightarrow 0} 6x^4 + x^3 + 2 = 6(0)^4 + (0)^3 + 2 = 2$.

   c) $\lim_{x\longrightarrow 0^-} f(x) = \lim_{x\longrightarrow 0^-} 3x^2 + 2x^3 + 7 = \lim_{x\longrightarrow 0} 3x^2 + 2x^3 + 7 = 7$ by part a).

   d) $\lim_{x\longrightarrow 0^+} f(x) = \lim_{x\longrightarrow 0^+} 6x^4 + x^3 + 2 = \lim_{x\longrightarrow 0} 6x^4 + x^3 + 2 = 2$ by part b).

   e) $f(x)$ is continuous at all a except for $a = 0$. Reasons: For all numbers a, except for $a = 0$, the function $f(x)$ can be thought of as a polynomial for x close to a. Since $\lim_{x\longrightarrow 0^-} f(x)$ and $\lim_{x\longrightarrow 0^+} f(x)$ are not the same, the limit $\lim_{x\longrightarrow 0} f(x)$ does not exist. Thus $f(x)$ is not continuous at $x = 0$.

   f) Since polynomials are differentiable everywhere, $f'(a)$ exists for all a except possibly $a = 0$. Since $f(x)$ is not continuous at $a = 0$, $f'(0)$ does not exist.

3. $f'(x) = x^2 - 6x + 8$.

a) We calculate the difference quotient for $f'(x)$:

$$\begin{aligned}[f'(a+h)-f'(a)]/h &= [((a+h)^2-6(a+h)+8)-(a^2-6a+8)]/h \\ &= [((a^2+2ah+h^2)-6(a+h)+8)-(a^2-6a+8)]/h \\ &= (2ah+h^2-6h)/h \\ &= (h(2a+h-6))/h \\ &= 2a+h-6.\end{aligned}$$

Therefore $f''(a) = \lim_{h\to 0}[f'(a+h)-f'(a)]/h = \lim_{h\to 0}(2a+h-6) = 2a-6$ from which we conclude that $f''(x) = 2x-6$. **Note: It is not correct to write** $2a+h-6) = 2a-6$ **to conclude the calculation of** $f''(a)$.

b) Intervals on which the graph of $y = f(x)$ is concave down; that is the intervals on which $f''(x) < 0$; answer $(-\infty, 3)$ by part a).

c) Intervals on which the $f'(x)$ is increasing; that is the intervals on which $f''(x) = (f')'(x) > 0$; answer $(3, \infty)$ by part a).

d) Intervals on which the graph of $y = f(x)$ is concave up; that is the intervals on which $f''(x) > 0$; answer $(3, \infty)$ from part c).

e) Intervals on which the $f(x)$ is increasing; that is the intervals on which $f'(x) > 0$; thus, as $f'(x) = x^2-6x+8 = (x-2)(x-4)$ describes a parabola which opens upward and crosses the x-axis at $x = 2, 4$, these intervals are $(-\infty, 2)$ and $(4, \infty)$.

4. $s(t) = t^3 - 4t^2 - 3t$ describes the position of a body moving along a straight line at time $t \geq 0$. The instantaneous velocity of the body at time t is given to be $v(t) = s'(t) = 3t^2 - 8t - 3$.

a) By algebra, see part a) of problem 3, $a(t) = v'(t) = 6t - 8$.

b) The body is moving to the right, that is $s(t)$ is increasing, which is to say that $v(t) = s'(t) = (3t+1)(t-3) > 0$, when $t > 3$. The body is moving to the left, that is when $s(t)$ is decreasing or $v(t) < 0$, when $0 \leq t < 3$.

c) The velocity of the body is increasing, that is $v'(t) = a(t) = 6t-8 > 0$, when $4/3 < t$. The velocity of the body is decreasing, that is $v'(t) = a(t) = 6t-8 < 0$, when $0 \leq t < 4/3$.

d) Origin of motion is $s(0) = 0$. Solving $s(t) = 0$, or $t(t^2-4t-3) = 0$, yields $t = 0$ and $t = (4+\sqrt{28})/2 = 2+\sqrt{7}$; the latter follows by the quadratic formula.

5. Suppose that $f(x)$ is defined on all real numbers and $f'(x)$ exists for all real numbers x. Below is a table of values of the function.

x	-1	0	2	5	7
f(x)	3	4.5	5.4	6.0	6.2

(1)

a) Use the table above to complete the table below which records average rates of change of $f(x)$ on the interval $[-1,0]$, $[0,2]$, ...

interval	-1 – 0	0 – 2	2–5	5–7
average rate of change	1.5	.45	.2	.1

b) Use the table (1) to complete the table below by estimating the instantaneous rate of change of $f(x)$ at the indicated values of x.

x	-1	0	2	5	7
f′(x)	1.5	.975	.325	.15	.1

c) Use the table constructed in part b) to complete the table below by estimating the second derivative.

x	-1	0	2	5	7
f″(x)	$-.2625$	$-.2938$	$-.4547$	$-.3042$	$-.025$

The entries are rounded to 4 decimal places.

6. Let $f(x) = x + 1/(1+x^2)$. Then $f'(x) = 1 - (2x/(1+x^2)^2)$.

a) The average rate of change of $f(x)$ on the interval $[-1,2]$ is $(f(2) - f(-1))/(2 - (-1)) = (11/5 - (-1/2))/3 = 27/30 = .9$.

b) The instantaneous rate of change of $f(x)$ at $x = -3$ is $f'(-3) = 1+6/10^2 = 106/100 = 1.06$.

c) The derivative of $f(x)$ at $x = 7$ is $f'(7) = 1 - 14/50^2 = 2486/2500 = .9944$.

d) The slope of the line tangent to the graph of $y = f(x)$ at the point $(1, 3/2)$ is $f'(1) = 1 - 2/2^2 = 1/2$.

e) An equation of the tangent line to the graph of $y = f(x)$ at $x = 4$ is, working from $y - f(4) = f'(4)(x-4)$, is $y - 69/17 = (281/289)(x-4)$.

f) Using tangent line approximation to estimate $f(4.03)$, we use part c) to calculate $y = (281/289)(4.03-4) + 69/17 = (281/289)(3/100) + 69/17$.

g) Assume that $f(x)$ is the cost in dollars of producing x items, $x > 0$. Using the derivative to estimate the cost of producing item number 101 yields $f'(100) = 1 - 200/10001^2$.

7. Sample Hour Exam I (S. Smith's version), problem 2), addendum.

a) $P(t) = P_0^{kt}$, where $k = \ln(1223.67/1123.96) = .0849963762$ (to 10 decimal places). Thus the continuous growth rate is 08.49963762%, using thus approximation.

b) We may also write $P(t) = P_0 a^t$, where a is the base, which is $P(5)/P(4) = 1223.67/1123.96 = 1.088713121$ (to 9 decimal places). Since $a = 1.088713121 = 1 + .088713121$, the annual growth rate is 08.8713121%, using this approximation.

c) Time t to double satisfies $P(t) = 2P_0$, or $2P_0 = P_0 e^{kt}$. Thus $2 = e^{kt}$ which means that $t = \ln 2/k = \ln 2/\ln(1223.67/1123.96) = 8.155020381$.

Warning: Do not round your answers to be used in subsequent calculations to severely. This may throw subsequent answers way off. Rounding to 4 decimal places is ok.