Math 180 Fall 2004 Final Examination Solution

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1. (**15 points total**) Find the derivative of each of the following functions. Do NOT simplify after taking derivatives.

   (a) $\ln(e^x + 1)$ (b) $e^{2x}\sinh(3x)$ (c) $\dfrac{x^2}{x+2}$

   Solution: (a) $\dfrac{e^x}{e^x+1}$ (**5 points**) (b) $2e^{2x}\sinh(3x) + 3e^{2x}\cosh(3x)$ (**5 points**)
   (c) $\dfrac{2x(x+2) - x^2}{(x+2)^2}$ (**5 points**)

2. (**25 points total**) A curve is given by the equation $y^2 + 2y = x^3 - 4x$.

   (a) Use implicit differentiation to find a formula for $\dfrac{dy}{dx}$.
   Solution: $2yy' + 2y' = 3x^2 - 4$, so $y'(2y+2) = 3x^2 - 4$. Thus $y' = (3x^2 - 4)/(2y+2)$. (**10 points**)

   (b) Find the equation of the tangent line to this curve at the point $(2, -2)$.
   Solution: At $(2,-2)$ the slope is $(3 * 4 - 4)/(-4+2) = -4$. Tangent line is: $(y+2)/(x-2) = -4$ or $y = -4x + 6$. (**10 points**)

   (c) Use part (b) to find the approximate value of y when $x = 1.98$.
   Solution: If $x = 1.98$, then y is approximately $-4(1.98) + 6 = -1.92$. Or using the linear approximation: $-2 + -4(1.98 - 2) = -1.92$. (**5 points**)

3. (**25 points total**) Let $f(x) = x^2 e^{-2x}$. *You must use calculus and show all your work in this problem.*

   (a) Find and classify all critical points of f (as to local maximum, local minimum, or neither)
   Solution: $f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2xe^{-2x}(1-x)$. Critical points are $f'(x) = 0$, so $x = 0$ and $x = 1$ are the only critical points. At $x = 0$ the value of f' changes from $-$ to $+$ so $x = 0$ (or (0,0)) is a local minimum. At $x = 1$ the value of f' changes from $+$ to $-$ so $x = 1$ or $(1, e^{-2})$ is a local maximum. Can also be done using the second derivative test. (**15 points**)

   (b) Find the global maximum and global minimum of f for $-1 \le x \le 2$.
   Solution: $f(0) = 0$, $f(1) = e^{-2} \approx .135$, $f(-1) = e^2$ and $f(2) \approx .073$. So $e^2 \approx 7.39$ is the global max and 0 is the global minimum. (**10 points**)

4. (a) (**10 points**) If a is a constant, use algebra to find $\lim_{h\to 0} \frac{(a+h)^2 + 2(a+h) - a^2 - 2a}{h}$.
   Solution: The fraction simplifies to $(2ah + h^2 + 2h)/h = 2a + h + 2$. As $h \to 0$ this becomes $2a + 2$.

   (b) (**10 points**) Use L'Hôpital's Rule to find $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$.
   Solution: At $x = 0$ we have $0/0$. L'Hopital gives $2\sin(2x)/2x = \sin(2x)/x$. Again we have $0/0$ at $x = 0$ so use L'Hopital again to get $2\cos(2x)/1$. The limit as $x \to 0$ of this is 2.

5. (**20 points total**) Let $f(x) = \begin{cases} 3 - x & \text{if } x < 3; \\ x^2 + x + b & \text{if } x \ge 3. \end{cases}$

   (i) Find a value for b so that $f(x)$ is a continuous function.

Solution: We need $3 - 3 = 3^2 + 3 + b$. So $b = -12$. (**10 points**)

(ii) If b is as in part (i), is $f(x)$ differentiable for all x? Justify your answer.
Solution: Must have $(3-x)' = (x^2+x=12)'$ at $x = 3$. Then $-1 = 2*3+1$, which is impossible. So answer is NO. Other justifications: graph has sharp corner at $x =$, slopes of the tangent lines to $3-x$ and x^2+x+12 are different. (**10 points**)

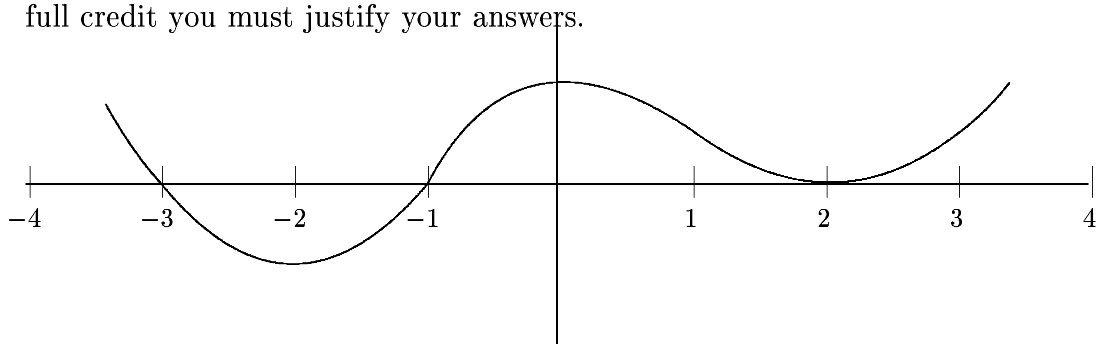
6. (**25 points total**) Consider the following table of values for the function $y = f(x)$.

(a) Estimate $\int_0^3 f(x)\,dx$ using the Left-hand Sum for $a = 0$, $b = 3$, and $n = 3$ rectangles;
Solution: $\Delta x = (b-a)/n = (3-0)/3 = 1$. Thus $LS = (2+3+4)*1 = 9$. (**12 points**)

(b) Estimate $\int_{.5}^3 f(x)\,dx$ using the Right-hand Sum for $a = 0.5$, $b = 3$, and $n = 5$ rectangles.
Solution: $\Delta x = (b-a)/n = (3-.5)/5 = .5$. Thus $RS = (3+7+4-11+110)(.5) = 113/2 = 56.5$. (**13 points**)

t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
f(t)	2	3	1	-4	3	10	7	6	4	3	-11	98	110

7. (**25 points total**) The graph below is a graph of f' (NOT the graph of f). Use the graph of f' to answer the following questions about the function f. To receive full credit you must justify your answers.



(a) Write down all x-values of critical points of f.

Solution: Critical points are where $f'(x) = 0$. Answer: -3, -1, 2. (**5 points**)

(b) Write down all x-values for which $f''(x) = 0$.

Solution: $f''(x) = 0$ at critical point for $f'(x)$, where tangent line to f' is has slope 0. Answer: -2, 0, 2. (**5 points**)

(c) For each critical point, indicate whether f has a local maximum, a local minimum, or neither at that point.

Solution: f' changes from $+$ to $-$ at $x = -3$, so $x = 3$ is a local max. f' changes from $-$ to $+$ at $x = -1$, so $x = -1$ is local min. f' does not change sign at $x = 2$, so neither local max nor local min. (**5 points**)

(d) For each value of x for which $f''(x) = 0$, indicate whether it is an inflection point of f.

Solution: All three of $-2,\ 0,\ 2$ are inflection points since f' changes from an increasing function to a decreasing function or from a decreasing function to an increasing function at each point. (**5 points**)

(e) For which value of x is $f(x)$ decreasing most rapidly.

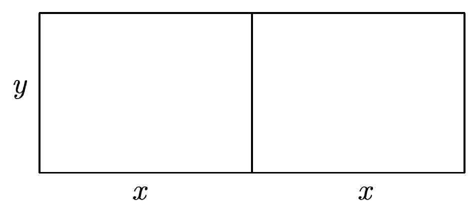
Solution: f' is the most negative at $x = -2$. (**5 points**)

8. (**25 points total**) A farmer has 1,200 feet of fencing and she wants use this fencing to form a rectangular pen consisting of two sections sharing a common side as in the diagram. Let x and y be the lengths as shown in the diagram.

(a) Express the area of the pen as a function of x.
Solution: $A = 2xy$ and $y = (1200 - 4x)/3$. So $A(x) = 2x(1200 - 4x)/3 = 800x - 8x^2/3$. (**5 points**)

(b) If x and y are as in the diagram, what values of x and y give the maximum total area of the pen?
Solution: $A'(x) = 800 - 16x/3$. So $A'(x) = 0$ at $x = 3 * 800/16 = 150$ ft. This is a global maximum since $A = 0$ when $x = 0$ or $x = 1200/4$ (since $y = 0$ in that case). If $x = 150$ then $y = 600/3 = 200$ ft. (**15 points**)

(c) What is the maximum area that the pen can have?
Solution: $A = 2 * 150 * 200 = 60,000$ square feet. (**5 points**)



9. (**20 points total**) Let $f(x) = 3x^2 + 2x$.

(a) Write a definite integral whose value is the area of the region that is bounded on the left by the vertical line $x = 1$, bounded on the right by the vertical line $x = \pi$, bounded above by the curve $y = f(x)$ and bounded below by the x-axis.

Solution:$\int_1^\pi (3x^2 + 2x)\, dx$. (**5 points**)

(b) Find the *exact* value of the area of the region described in part (a). (Your answer should involve π.)

Solution: $\int_1^\pi (3x^2 + 2x) = \pi^3 + \pi^2 - (1^3 + 1^2) = \pi^3 + \pi^2 - 2$. (**10 points**)

(c) Find the average value of $y = f(x)$ on the interval $[1, \pi]$.

Solution: $\dfrac{1}{\pi - 1} \displaystyle\int_1^\pi f(x)\, dx = (\pi^3 + \pi^2 - 2)/(\pi - 1)$, (**5 points**) or $\pi^2 + 2\pi + 2 \approx 18.153$.