MATH 180 Hour Exam II Solution **Radford** **11/12/04**

1. (20 points total)

a) $(x^7 + 7^x + \frac{3}{\sqrt{x}})' = (x^7 + 7^x + 3x^{-1/2})' = \boxed{7x^6 + 7^x \ln 7 - \frac{3}{2}x^{-3/2} \text{ } (\mathbf{5\ points})}$

b)

$$\left((x^{20}+1)^{30}(x^{1/3}+\cos x+1)\right)'$$

$$= \boxed{\left(30(x^{20}+1)^{29}(20x^{19})\right)(x^{1/3}+\cos x+1)+(x^{20}+1)^{30}\left(\frac{1}{3}x^{-2/3}-\sin x\right) \text{ } (\mathbf{5\ points})}$$

c) $(\sinh x + \ln(\tan x) + \sin(x^2))' = \boxed{\cosh x + \frac{\sec^2 x}{\tan x} + 2x\cos(x^2) \text{ } (\mathbf{5\ points})}$

d)

$$\left(\frac{e^{\tanh x}+x^9-5}{4x^3-11x^5}\right)'$$

$$= \boxed{\frac{(e^{\tanh x}\mathrm{sech}^2\, x+9x^8)(4x^3-11x^5)-(e^{\tanh x}+x^9-5)(12x^2-55x^4)}{(4x^3-11x^5)^2} \text{ } (\mathbf{5\ points})}$$

2. (15 points total) Since $y = f(x)$ satisfies $x^3y^2+4y^5+x^3 = 20y$, by implicit differentiation $(3x^2)y^2 + (x^3)(2yy') + 4(5y^4y') + 3x^2 = 20y'$ and thus

a) $\frac{dy}{dx} = y' = \boxed{-\frac{3x^2y^2+3x^2}{2x^3y+20y^4-20} \text{ } (\mathbf{6\ points})}$.

b) At the point $(2,1)$ the slope of the tangent line is $\boxed{-\frac{3}{2} \text{ } (\mathbf{3\ points})}$. Thus

$\boxed{y-1 = -\frac{3}{2}(x-2) \text{ } (\mathbf{3\ points})}$, or $y = -\frac{3}{2}(x-2)+1$, or $y = -\frac{3}{2}x+4$.

c) $f(2.25) \approx -\frac{3}{2}(2.25-2)+1 = -\frac{3}{8}+1 = \boxed{\frac{5}{8} \text{ } (\mathbf{3\ points})}$.

3. (18 points total) Since $f(x) = 6x^2 - x^4 = x^2(6-x^2)$, we have $f'(x) = 12x - 4x^3 = 4x(3-x^2)$ and $f''(x) = 12 - 12x^2 = 12(1-x^2)$.

a) The *critical points* of $y = f(x)$ are $\boxed{x = -\sqrt{3}, 0, \sqrt{3}. \ (\mathbf{3\ points})}$

b) Since $f'(-2) > 0$, $f'(-1) < 0$, $f'(1) > 0$, and $f'(2) < 0$, the intervals on which $f(x)$ is $\boxed{\textit{increasing} \text{ are } (-\infty, -\sqrt{3}],\ [0, \sqrt{3}]}$, on which $f(x)$ is $\boxed{\textit{decreasing} \text{ are } [-\sqrt{3}, 0],\ [\sqrt{3}, \infty)}$. (**3 points**)

c) Since $f''(-2) < 0$, $f''(0) > 0$, and $f''(2) < 0$, the interval(s) on which the graph of $y = f(x)$ is $\boxed{\textit{concave down} \text{ are } (-\infty, -1],\ [1, \infty)}$, on which the graph is $\boxed{\textit{concave up} \text{ is } [-1, 1]}$. (**3 points**)

d) The *inflection points* on the graph of $y = f(x)$ are $\boxed{(-1, 5),\ (1, 5)\ (\mathbf{3\ points})}$.

e) Local maxima $(-\sqrt{3}, 9)$, $(\sqrt{3}, 9)$; local minimum $(0, 0)$; inflection points $(-1, 5)$, $(1, 5)$; graph crosses the x-axis at $x = -\sqrt{6}, \sqrt{6}$ (**3 points for points, 3 for shape of graph**). Graph shape is roughly an upside down "W".

4. (17 points total) $C = 15x + 5y + 10x + 7y = 25x + 12y$. Since $xy = 1200$ we have

a) $\boxed{C(x) = 25x + 12\left(\frac{1200}{x}\right)\ (\mathbf{6\ points})}$, where $\boxed{x > 0\ (\mathbf{1\ point})}$.

b) $C'(x) = 25 - \frac{12 \cdot 1200}{x^2}$. Thus $C'(x) = 0$ has solutions $x = -24, 24$. Thus there is only one critical point for $C(x)$, namely $x = 24$, since $x > 0$. Now $C'(1) < 0$ and $C'(25) > 0$. Therefore $C(x)$ has a minimum value which is $C(24)$. (**3 points for justification of minumum**) Dimensions: $\boxed{x = 24 \text{ and } y = \frac{1200}{24} = 50\ (\mathbf{5\ points})}$

c) $\boxed{C(24) = 25 \cdot 24 + 12 \cdot 50 = 1200 \text{ dollars } (\mathbf{2\ points})}$.

5. (15 points total) Using L'Hopital's rule:

a) $\boxed{\lim_{x \to \infty} \frac{8x^2 + 13x - 6}{11x^2 - 5x + 4} = \lim_{x \to \infty} \frac{16x + 13}{22x - 5} = \lim_{x \to \infty} \frac{16}{22} = \frac{8}{11}\ (\mathbf{9\ points})}$

b) $\boxed{\lim_{x \to 0} \frac{e^{3x} - 1}{7x} = \lim_{x \to 0} \frac{3e^{3x}}{7} = \frac{3}{7}.\ (\mathbf{6\ points})}$

6. (15 points total) Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}$ and $\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}\ (\mathbf{5\ points})}$.

Since $\frac{dV}{dt} = 6$ cubic feet per minute, when $r = 4$ feet we have $\boxed{6 = 4\pi \cdot 4^2 \frac{dr}{dt}\ (\mathbf{5\ points})}$ so $\boxed{\frac{dr}{dt} = \frac{6}{4^3\pi} \approx .0298 \text{ feet per minute. } (\mathbf{5\ points})}$