1. **20 points total** We first note that $n^3 < -2n^2 + 15n$ is equivalent to $n(n+5)(n-3) = n^3 + 2n - 15n < 0$.

(a) Since $0 < n < 3$, and is an integer, $n = 1, 2$.

Case 1: $n = 1$. Here $n(n+5)(n-3) = 1(6)(-2) = -12 < 0$. (**3 points**)

Case 2: $n = 2$. Here $n(n+5)(n-3) = 2(7)(-1) = -14 < 0$. (**3 points**)

(b) A "working backwards" solution is:

$$\begin{aligned}
&n^3 < -2n^2 + 15n\\
&n^3 + 2n - 15n < 0\\
&n(n+5)(n-3) < 0\\
&n, n+5 > 0, n-3 < 0\\
&n > 0, n-3 < 0\\
&0 < n < 3.\ \ (\textbf{8 points})
\end{aligned}$$

(c) We have noted that $n^3 < -2n^2 + 15n$ is equivalent to $n(n+5)(n-3) < 0$. The latter is not the case if one of the factors is zero; that is if $n = 0, -5$, or 3. Recalling a bit of calculus this suggests the various other cases.

Case 1: $n < -5$. Here $n < 0$, $n+5 < 0$, and $n-3 < 0$. Therefore the expression $n(n+5)(n-3)$ is negative.

Case 2: $-5 < n < 0$. Here $n < 0$, $n+5 > 0$, and $n-3 < 0$. Therefore the expression $n(n+5)(n-3)$ is positive.

Case 3: $0 < n < 3$. Here $n > 0$, $n+5 > 0$, and $n-3 < 0$. Therefore the expression $n(n+5)(n-3)$ is negative.

Case 4: $3 < n$. Here $n > 0$, $n+5 > 0$, $n-3 > 0$. Therefore the expression $n(n+5)(n-3)$ is positive.

Combining the results of the various cases we see that $n^3 < -2n^2 + 15n$ exactly when $n < -5$ or $0 < n < 3$. (**6 points**)

2. **20 points total** Note that $a^2 \geq 7a$ is equivalent to $a(a-7) = a^2 - 7a \geq 0$.

(a) Suppose the hypothesis $a^2 \geq 7a$ is true and the conclusion $a \leq 0$ or $a \geq 7$ is false, that is $0 < a < 7$. Since $a > 0$ and $a < 7$ necessarily $a^2 < 7a$, a direct contradiction of the hypothesis. Therefore the conclusion must be true. (**6 points**)

(b) We have noted that $a^2 \geq 7a$ is equivalent to $a(a-7) \geq 0$. This is equivalent one of three cases:

Case 1: $a = 0$ or $a = 7$. In either case $a \leq 0$ or $7 \leq a$. (**2 points**)

Case 2: $a, a - 7 > 0$. This is the same as $a > 7$. Thus $a \leq 0$ or $7 \leq a$. (**3 points**)

Case 3: $a, a - 7 < 0$. This is the same as $a < 0$. Thus $a \leq 0$ or $7 \leq a$. (**3 points**)

(c) "$0 < a < 7$ implies $a^2 < 7a$"; the translation of "not ($a \leq 0$ or $a \geq 7$) implies not ($a^2 \geq 7a$)". (**6 points**)

3. **20 points total** Note that $a^2 - 12a + 35 < 0$ is equivalent to $(a - 5)(a - 7) < 0$.

(a) Suppose the hypothesis $a^2 - 12a + 35 < 0$ is true and the conclusion $5 \leq a < 7$ is false, that is $a < 5$ or $a \geq 7$.

Case 1: $a < 5$. Then $a - 5, a - 7 < 0$ which means that $(a - 5)(a - 7) > 0$, a contradiction. (**4 points**)

Case 2: $a \geq 7$. Then $a - 5 > 0$, $a - 7 \geq 0$ which means that $(a - 5)(a - 7) \geq 0$, a contradiction. (**4 points**)

Therefore the conclusion must be true.

(b) We have noted that $a^2 - 12a + 35 < 0$ is equivalent to $(a - 5)(a - 7) < 0$. Thus one of the two factors must be positive and the other negative.

Case 1: $a - 5 > 0$, $a - 7 < 0$. Thus $5 < a < 7$ which means $5 < a \leq 7$. (**3 points**)

Case 2: $a - 5 < 0$, $a - 7 > 0$. Thus $a < 5$ and $7 < a$. This case is not possible. (**3 points**)

(c) The converse is "$5 \leq a < 7$ implies $a^2 - 12a + 35 < 0$". This false. Take $a = 5$. Then $a^2 - 12a + 35 = 0$. (**6 points**)

4. **20 points total** An integer n is even if $n = 2m$ for some integer m and is odd if $n = 2m + 1$ for some integer m.

(a) By cases.

Case 1: Both n, n' even. Then $n = 2m$ and $n' = 2m'$ for some integers m, m'. Thus $n + n' = 2m + 2m' = 2(m + m')$ is even. (**3 points**)

Case 2: Both n, n' odd. Then $n = 2m + 1$ and $n' = 2m' + 1$ for some integers m, m'. Thus $n + n' = (2m + 1) + (2m' + 1) = 2(m + m' + 1)$ is even. (**3 points**)

Case 3: n is even and n' odd. Then $n = 2m$ and $n' = 2m' + 1$ for some integers m, m'. Thus $n + n' = 2m + (2m' + 1) = 2(m + m') + 1$ is odd. (**3 points**)

Case 4: n odd and n' even. Then $n + n' = n' + n$ is odd by Case 3. (**1 points**)

(b) By cases.

Case 1: n even. Then $n = 2m$ for some integer m. Therefore

$$n^2 + 5n = (2m)^2 + 5(2m) = 2(2m^2 + 5m)$$

is even. (**5 points**)

Case 2: n odd. Then $n = 2m + 1$ for some integer m. Therefore

$$n^2 + 5n = (2m + 1)^2 + 5(2m + 1) = (4m^2 + 4m + 1) + 5(2m + 1) = 2(2m^2 + 7m + 1)$$

is even. (**5 points**)

5. **20 points total** Here is a formal proof. Let $P(n)$ be the statement "$n^2 + 5n$ is even", $n \geq 1$. P(1) is true since $n^2 + 5n = 6$ is even. (**6 points**)

Suppose that $n \geq 1$ and P(n) is true. Then $n^2 + 5n = 2m$ for some integer m. We will show that P(n+1), that is "$(n + 1)^2 + 5(n + 1)$ is even", is true. The calculation

$$(n + 1)^2 + 5(n + 1) = (n^2 + 2n + 1) + (5n + 5) = (n^2 + 5n) + 2(n + 3) = 2(m + n + 3)$$

shows that P(n+1) is true. We have shown that P(n) implies P(n+1) for all $n \geq 1$. (**10 points**). Therefore P(n) is true for all $n \geq 1$ by induction. (**4 points**)