Math 215 **Summer 2008** **Radford**

Written Homework # 5

Due at the beginning of class 07/18/08

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1. We define the Cartesian product of sets $A_1, \ldots, , A_n$ inductively by

$$A_1 \times \cdots \times A_n = \begin{cases} A_1 & : \quad n = 1; \\ (A_1 \times \cdots \times A_{n-1}) \times A_n & : \quad n > 1. \end{cases}$$

Suppose that $A_1, \ldots, A_n$ are finite sets. Show, by induction, that $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$ for $n \geq 1$. (You may assume this is the case for $n = 2$.)

2. What are the logical relationships between the following statements:

(a) $\exists\, a \in A, \exists\, b \in B$, not $P(a, b)$;

(b) $\forall\, a \in A, \forall\, b \in B,\ P(a, b)$;

(c) $\forall\, a \in A, \exists\, b \in B,\ P(a, b)$;

(d) $\exists\, a \in A, \forall\, b \in B$, not $P(a, b)$?

Comment: To show that one statement does not always imply another, supply a specific counterexample. Try $P(a, b) : a^2 = 5$ or $P(a, b) : b^3 = 7$ for example, where $A = B = \mathbf{R}$.

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, b \in \mathbf{R}$. The very compact definition of $\lim_{x \longrightarrow a} f(x) = b$ is

$$\forall\, \epsilon > 0, \exists\, \delta > 0, P(\epsilon, \delta),$$

where

$$P(\epsilon, \delta) : \forall\, x \in \mathbf{R}, (0 < |x - a| < \delta) \Rightarrow (|f(x) - b| < \epsilon).$$

3. Let $f(x) = \begin{cases} 3x + 5 & : \quad x \neq 4 \\ -10 & : \quad x = 4. \end{cases}$ Prove that $\lim_{x \longrightarrow 4} f(x) = 17$.

4. We examine what it means for $\lim_{x \longrightarrow a} f(x)$ not to exist.

(a) Using the compact definition of $\lim_{x \longrightarrow a} f(x) = b$, express in the same manner "not ($\lim_{x \longrightarrow a} f(x) = b$)". Write "not $P(\epsilon, \delta)$" explicitly.

(b) Show that $\lim_{x \longrightarrow 0} f(x) = b$ is false for all $b \in \mathbf{R}$, where $f(x) = \begin{cases} 1 & : \quad x \geq 0 \\ -1 & : \quad x < 0. \end{cases}$

5. Let $f : X \longrightarrow Y$ be a function. Then f is surjective means $\forall\, y \in Y, \exists\, x \in X,\ f(x) = y$.

(a) In the style of the definition of surjective, express what it means for f *not* to be surjective.

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $f(x) = x^2 - 2x + 20$. Show that $f(x)$ is not surjective by showing that the condition of part (a) is satisfied.