

# The Epsilon-Delta Definition of Limit of a Function

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Suppose that  $f : \mathbf{R} \longrightarrow \mathbf{R}$  is a function and  $a, L \in \mathbf{R}$ . Then

$$\lim_{x \rightarrow a} f(x) = L$$

means for all positive real numbers  $\epsilon$  there exists a positive real number  $\delta$  such that  $0 < |x - a| < \delta$  implies  $|f(x) - L| < \epsilon$ . This is the *epsilon-delta* definition of the limit of the function  $y = f(x)$  at  $x = a$ . We can reformulate the definition in terms of quantifiers as

$$\forall \epsilon > 0, (\exists \delta > 0, (\forall x \in (a - \delta, a + \delta) - \{a\}, |f(x) - L| < \epsilon)).$$

Usually  $\forall \epsilon > 0$  is shorthand for  $\forall \epsilon \in \mathbf{R}^+$  and  $\exists \delta > 0$  is shorthand for  $\exists \delta \in \mathbf{R}^+$ . We note that the definition of limit of function is more complicated when the domain of  $f(x)$  is not all of  $\mathbf{R}$ .

The negation of  $\forall a \in A, P(a)$  is

not $(\forall a \in A, P(a))$	which is equivalent to	$\exists a \in A, \text{not } P(a)$ .
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Thus the negation of  $\lim_{x \rightarrow a} f(x) = L$  is

$$\text{not } (\forall \epsilon > 0, (\exists \delta > 0, (\forall x \in (a - \delta, a + \delta) - \{a\}, |f(x) - L| < \epsilon))),$$

or

$$\exists \epsilon > 0, \text{not } (\exists \delta > 0, (\forall x \in (a - \delta, a + \delta) - \{a\}, |f(x) - L| < \epsilon)),$$

or

$$\exists \epsilon > 0, (\forall \delta > 0, \text{not } (\forall x \in (a - \delta, a + \delta) - \{a\}, |f(x) - L| < \epsilon)),$$

or

$$\exists \epsilon > 0, (\forall \delta > 0, (\exists x \in (a - \delta, a + \delta) - \{a\}, \text{not } (|f(x) - L| < \epsilon))),$$

or

$$\exists \epsilon > 0, (\forall \delta > 0, (\exists x \in (a - \delta, a + \delta) - \{a\}, |f(x) - L| \geq \epsilon)).$$

In words, to say that  $\lim_{x \rightarrow a} f(x) = L$  false is the same as saying there exists a positive number  $\epsilon$  such that for all positive numbers  $\delta$  there exists an  $x \in \mathbf{R}$  such that  $0 < |x - a| < \delta$  and  $|f(x) - L| \geq \epsilon$ .

Problem 2 of Homework #6 establishes the triangle inequality which is

$$|a + b| \leq |a| + |b|$$

for all  $a, b \in \mathbf{R}$ . The triangle inequality is useful in computations, although it is not needed for Problem 3.

We end with an epsilon–delta proof which is typical in form but more complicated than ones needed for Problem 3.

Let  $f(x) = x^2$  for all  $x \in \mathbf{R}$ . We will show that

$$\lim_{x \rightarrow a} f(x) = a^2.$$

We use the fact that  $|bc| = |b||c|$  for all  $b, c \in \mathbf{R}$  and we will use the triangle inequality twice in our calculations.

Let  $\epsilon > 0$ . We want to find  $\delta > 0$  such that  $0 < |x - a| < \delta$  implies  $|f(x) - a^2| = |x^2 - a^2| < \epsilon$ .

Let  $x \in \mathbf{R}$ . Since

$$\begin{aligned} |f(x) - a^2| &= |x^2 - a^2| \\ &= |(x - a)(x + a)| \\ &= |x - a||x + a| \\ &\leq |x - a|(|x| + |a|) \\ &= |x - a|(|a + (x - a)| + |a|) \\ &\leq |x - a|(|a| + |x - a| + |a|) \\ &= |x - a|(2|a| + |x - a|) \\ &\leq |x - a|(2|a| + 1) \quad \text{***** if } |x - a| \leq 1 \text{ *****} \\ &< \epsilon, \end{aligned}$$

if  $|x - a| < \frac{\epsilon}{(2|a| + 1)}$  also. Thus  $|f(x) - a^2| < \epsilon$  if

$$0 < \delta < 1, \frac{\epsilon}{(2|a| + 1)}.$$

Since both 1 and the quotient are positive, there is such a  $\delta$ . For example let  $\delta$  be one half of the minimum of 1 and the quotient.