Math 215, Fall 05 Homework #4 Solution

09/21/05

1. (**20 points total**) The assertion may be reformulated: $(a-1)(a-5) \geq 0$ implies $a \leq 1$ or $5 \leq a$.

a) Suppose that the hypothesis $(a-1)(a-5) \geq 0$ is true and the conclusion $a \leq 1$ or $5 \leq a$ is false. Then $1 < a < 5$. But then $a - 1 > 0$ and $a - 5 < 0$. Therefore $(a-1)(a-5) < 0$, a contradiction. We have shown that the hypothesis implies the conclusion; the assertion is true. (**5 points**)

b) Suppose that the hypothesis $(a-1)(a-5) \geq 0$ and is true and $a \not\leq 1$. Then $1 < a$. Therefore $a - 1$ is positive. Since $(a-1)(a-5) \geq 0$ either $a - 5$ is zero or positive. In any case $5 \leq a$.

We have shown $a \not\leq 1$ implies $5 \leq a$. Therefore $a \leq 1$ or $5 \leq a$. (**5 points**)

c) The contrapositive of the assertion is: $1 < a < 5$ implies $(a-1)(a-5) < 0$. As for a proof, assume that $1 < a < 5$. Then $a - 1$ is positive and $a - 5$ is negative. Therefore $(a-1)(a-5)$ is negative. (**5 points**)

d) The converse of the assertion is: $a \leq 1$ or $5 \leq a$ implies $(a-1)(a-5) \geq 0$.

Case 1: $a \leq 1$. Here $a - 5 < a - 1 \leq 0$. Therefore $a - 5$ is negative and $a - 1$ is zero, in which case $(a-1)(a-5)$ is zero, or $a - 1$ is also negative, in which case $(a-1)(a-5)$ is positive. In either case $(a-1)(a-5) \geq 0$.

Case 2: $5 \leq a$. Here $0 \leq a - 5 < a - 1$. Therefore $a - 1$ is positive and $a - 5$ is zero, in which case $(a-1)(a-5)$ is zero, or $a - 5$ is also positive, in which case $(a-1)(a-5)$ is positive. In either case $(a-1)(a-5) \geq 0$. (**5 points**)

2. (**20 points total**) For $a = 5$ we note that $a^2 - 6a + 5 = 25 - 30 + 5 = 0 \geq 0$. Therefore the assertion is true when $a = 5$. (**5 points**) Suppose $a \geq 5$ and the assertion is true. Then

$$(a+1)^2 - 6(a+1) + 5 \quad = \quad (a^2 + 2a + 1) - (6a + 6) + 5$$

$$
\begin{aligned}
&= (a^2 - 6a + 5) + (2a - 5) \\
&= (a^2 - 6a + 5) + 2(a - 5) + 5 \\
&\geq 0 + 2 \cdot 0 + 5 \\
&\geq 0.
\end{aligned}
$$

Thus if the assertion holds for $a \geq 5$ it holds for $a+1$. Therefore the assertion holds for all $a \geq 5$ by induction. (**5 points for induction step set-up, 10 points for calculations**)

3. (**20 points total**) Since $1^2 = 1 = \dfrac{1(1+1)(2+1)}{6}$ the formula is true for $n = 1$. (**5 points**) Suppose the formula is true for $n \geq 1$. Then

$$
\begin{aligned}
1^2 + \cdots + (n+1)^2 &= (1^2 + \cdots + n^2) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(2n+3)(n+2)}{6} \\
&= \frac{(n+1)(2(n+1)+1)((n+1)+1)}{6}
\end{aligned}
$$

which is equal to the right hand side of the formula with $n+1$ replacing n. Thus if the formula is true for n it is true for $n+1$. By induction the formula is true for all $n \geq 1$. (**5 points for induction step set-up, 10 points for calculations**)

4. (**20 points total**) Here we verify two familiar exponent laws for non-negative integers.

a) $a^{m+0} = a^m = a^m 1 = a^m a^0$ since $a^0 = 1$ by convention. Thus the exponent law holds for $n = 0$. (**3 points**) Suppose that $n \geq 0$ and the exponent law holds. Then

$$\begin{aligned} a^{m+(n+1)} &= a^{(m+n)+1} \\ &= a^{m+n}a \quad \text{by definition of powers of } a \\ &= (a^m a^n)a \quad \text{induction hypothesis} \\ &= a^m(a^n a) \\ &= a^m a^{n+1}. \quad \text{by definition of powers of } a \end{aligned}$$

Thus if the exponent law holds for $n \geq 0$ it holds for $n+1$. By induction the exponent law holds for all $n \geq 0$. (**2 points for induction step set-up, 5 points for calculations**)

b) $(a^m)^0 = 1 = a^0 = a^{m0}$ since $b^0 = 1$ for all real numbers b. (**3 points**) Suppose that $n \geq 0$ and the exponent law holds. Then

$$\begin{aligned} (a^m)^{n+1} &= (a^m)^n a^m \quad \text{by definition of powers of } b = a^m \\ &= a^{mn}a^m \quad \text{induction hypothesis} \\ &= a^{mn+m} \quad \text{by part a)} \\ &= a^{m(n+1)}. \end{aligned}$$

Thus if the exponent law holds for $n \geq 0$ it holds for $n+1$. By induction the exponent law holds for all $n \geq 0$. (**2 points for induction step set-up, 5 points for calculations**)