Math 215, Fall 05 Homework #5 Solution

10/14/05

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1. (**20 points total**) Many induction arguments come down to the case of $n = 2$. This problem gives such an example.

a) Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in C$ since $A \subseteq C$. If $x \in B$ then $x \in C$ since $B \subseteq C$. In any event (case) $x \in C$. We have shown $x \in A \cup B$ implies $x \in C$. By definition of subset $A \cup B \subseteq C$. (**8 points**)

b) We wish to prove the following assertion by induction on $n \geq 1$. Let $A_1, \ldots, A_n \subseteq C$. Then $A_1 \cup \cdots \cup A_n \subseteq C$.

The assertion is true for $n = 1$ since $A_1 \cup \cdots \cup A_n = A_1 \subseteq C$ in this case. (**4 points**) Suppose $n \geq 1$ and the assertion is true for n. Let $A_1, \ldots, A_{n+1} \subseteq C$. Set $A = A_1 \cup \cdots \cup A_n$ and $B = A_{n+1}$. Then $A \subseteq C$ since the assertion is true for n (the induction hypothesis) and $B \subseteq C$ by assumption. Since $n + 1 > 1$ we can usethe inductive definition of finite union and part a) to calculate

$$A_1 \cup \cdots \cup A_{n+1} = (A_1 \cup \cdots \cup A_n) \cup A_{n+1} = A \cup B \subseteq C.$$

Therefore the assertion is true for $n + 1$. By induction the assertion is true for all $n \geq 1$. (**8 points**)

2. (**20 points total**) Since S is very small in each case, to form the power set $P(S)$ we can first list the subsets of S.

a) $S = \emptyset$. The list of subsets of $\emptyset$ consists of one item, namely

$$\emptyset.$$

Thus

$$P(S) = \{\emptyset\}.$$

(**4 points**)

b) $S = \{1\}$. The list of subsets of $\{1\}$ is

$$\emptyset, \{1\}.$$

Therefore

$$P(S) = \{\emptyset, \{1\}\}.$$

(**4 points**)

c) $S = \{\emptyset\}$. In principle parts b) and c) are the same since in both cases S has one element. The list of subsets of $\{\emptyset\}$ is

$$\emptyset, \{\emptyset\}$$

and therefore

$$P(S) = \{\emptyset, \{\emptyset\}\}.$$

(**4 points**)

d) $S = \{8, 5\}$. The list of subsets of S is

$$\emptyset, \{8\}, \{5\}, \{8, 5\}.$$

Therefore

$$P(S) = \{\emptyset, \{8\}, \{5\}, \{8, 5\}\}.$$

(**4 points**)

e) $S = P(\{1\}) = \{\emptyset, \{1\}\}$ by part b). Here $\{1\}$ is an element of S. Thus the list of subsets of S is

$$\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}$$

and hence

$$P(S) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}.$$

(**4 points**)

3. (**20 points total**) The purpose of this exercise is to show the equivalence of De Morgan's Laws using the fact that $(A^c)^c = A$.

a) Since the first and last columns of the truth table

$x \in A$	$x \in A^c$	$x \in (A^c)^c$
T	F	T
F	T	F

are the same, the sets A and $(A^c)^c$ are the same. (**4 points**)

b) Suppose that $(A \cup B)^c = A^c \cap B^c$ for $A, B \in U$. Since $A^c, B^c \subseteq U$, we can use part a) and apply this law to A^c and B^c to obtain

$$(A \cap B)^c = ((A^c)^c \cap (B^c)^c)^c = ((A^c \cup B^c)^c)^c = A^c \cup B^c.$$

Therefore $(A \cap B)^c = A^c \cup B^c$ for $A, B \subseteq C$. (**8 points**)

Conversely, suppose that $(A \cap B)^c = A^c \cup B^c$ for $A, B \subseteq U$. Since $A^c, B^c \subseteq U$, we can use part a) and apply this law to A^c and B^c to obtain

$$(A \cup B)^c = ((A^c)^c \cup (B^c)^c)^c = ((A^c \cap B^c)^c)^c = A^c \cap B^c.$$

Therefore $(A \cup B)^c = A^c \cap B^c$ for $A, B \subseteq C$. (**8 points**)

4. (**20 points total**) Since the fourth and seventh columns of the truth table

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in (A \cup B)^c$	$x \in A^c$	$x \in B^c$	$x \in A^c \cap B^c$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

are the same, the sets $(A \cup B)^c$ and $A^c \cap B^c$ are the same.