Math 215, Fall 05 Homework #6

Due Friday, 10/14/05 at the beginning of class

You may assume the propositions and theorems of the text. You may assume that the product of two positive real numbers, and the product of two negative real numbers, is positive. You may also assume that the product of a positive real number and a negative real number is negative.

1. What are the logical relationships between the following statements? Justify your assertions.

a) $\forall a \in A$, not $P(a)$.

b) $\exists a \in A$, not $P(a)$.

c) not $(\forall a \in A,\ P(a))$.

2. The purpose of this exercise is to prove $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbf{R}$. Recall that

$$|a| = \begin{cases} a & : \quad a \geq 0; \\ -a & : \quad a < 0. \end{cases}$$

Thus $|a| \geq 0$ for all $a \in \mathbf{R}$. (You can assume this.)

a) Let $a, b > 0$. Show that $a^2 < b^2$ implies $a < b$. [Hint: Generally $b^2 - a^2 = (b-a)(b+a)$.]

b) Prove that $a \leq |a|$ and $a^2 = |a|^2$ for all $a \in \mathbf{R}$.

c) Show that $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbf{R}$. [Hint: Establish $|a+b|^2 = (a+b)^2 = \cdots \leq (|a|+|b|)^2$.]

3. Using the epsilon-delta definition of limit, prove the following assertions about limits:

a) $\lim_{x \longrightarrow a} f(x) = c$, where $f(x) = c$ is a constant function.

b) $\lim_{x \longrightarrow a} f(x) = ca + d$, where $f(x) = cx + d$ is a linear function.

4. Let A, B, C and D be sets. Part (iv) of Proposition 7.7.4 of the text states that

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$

and the text indicates that containment can not be replaced by equality always.

a) Show that if $A = C$ then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

b) Show that if $B = D$ then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

c) Find finite sets A and B such that $(A \times B) \cup (B \times A) \subset (A \cup B) \times (B \cup A)$.