Math 215, Fall 05 Homework #8 Solution

11/03/05

1. (**20 points total**) Here we show, in particular that there is a bijective correspondence between the set of all subsets of a (universal) non-empty set U and the set of all functions $f : U \longrightarrow \mathbf{R}$ with $\operatorname{Im} f \subseteq \{0, 1\}$.

a) Suppose that $A = B$. Then $\chi_A = \chi_B$. (**2 points**) Conversely, suppose that $\chi_A = \chi_B$. We will show that $A = B$. Let $x \in A$. Then $1 = \chi_A(x) = \chi_B(x)$ which means that $x \in B$ (otherwise $\chi_B(x) = 0$.) Therefore $x \in B$.

We have shown that $\chi_A = \chi_B$ implies $A \subseteq B$. Since we are assuming $\chi_A = \chi_B$, we have $A \subseteq B$ and $\chi_B = \chi_A$. By our previous argument $B \subseteq A$. Thus $A = B$. (**4 points**)

b) First $\chi_\emptyset = 0$. Let $x \in U$. Then $x \notin \emptyset$. Therefore $\chi_\emptyset(x) = 0$. WeE have shown $\chi_\emptyset(x) = 0$ for all $x \in U$. Therefore $\chi_\emptyset = 0$. (**3 points**)

Next $\chi_U = 1$. For all $x \in U$ we have $\chi_U(x) = 1$ by definition of the characteristic function. Therefore $\chi_U = 1$. (**3 points**)

c) Let $A = \{x \in U \mid f(x) = 1\}$. Let $x \in U$. If $x \in A$ then $f(x) = 1$ by definition of A. Suppose $x \notin A$. Then $f(x) \neq 1$ by definition of A. Since $\operatorname{Im} f \subseteq \{0, 1\}$ and $f(x) \neq 1$ necessarily $f(x) = 0$. Thus $f(x) = \chi_A(x)$ for all $x \in U$ which means that $f = \chi_A$. (**8 points**)

2. (**20 points total**) χ_{A^c} and $\chi_{A \cap B}$ are algebraic combinations of characteristic functions ($1 = \chi_U$).

a) Let $x \in U$. Since $x \in A$ if and only if $x \notin A^c$, and thus $x \in A^c$ if and only if $x \notin A$, the calculation

$$(1 - \chi_A)(x) = \begin{cases} 1 - 1 & : \ x \in A; \\ 1 - 0 & : \ x \notin A \end{cases} = \begin{cases} 0 & : \ x \notin A^c; \\ 1 & : \ x \in A^c \end{cases}$$

shows that $(1 - \chi_A)(x) = \chi_{A^c}(x)$ for all $x \in U$. Therefore $1 - \chi_A = \chi_{A^c}$. (**6 points**)

b) Let $x \in U$. Then

$$(\chi_A \chi_B)(x) = \chi_A(x)\chi_B(x) = \begin{cases} 1{\cdot}1 & : \; x \in A, x \in B; \\ 1{\cdot}0 & : \; x \in A, x \notin B; \\ 0{\cdot}1 & : \; x \notin A, x \in B; \\ 0{\cdot}0 & : \; x \notin A, x \notin B \end{cases} = \begin{cases} 1 & : \; x \in A, x \in B; \\ 0 & : \; x \in A, x \notin B; \\ 0 & : \; x \notin A, x \in B; \\ 0 & : \; x \notin A, x \notin B \end{cases}.$$

Since $x \in A \cap B$ if and only if $x \in A$ and $x \in B$, we conclude that

$$(\chi_A \chi_B)(x) = \begin{cases} 1 & : \; x \in A \cap B; \\ 0 & : \; x \notin A \cap B \end{cases} = \chi_{A \cap B}(x)$$

for all $x \in U$. Thus $\chi_A \chi_B = \chi_{A \cap B}$. (**6 points**)

c) Here is a solution which uses parts a) and b).

$$\begin{aligned} \chi_{A \cup B} &= \chi_{(A^c \cap B^c)^c} \\ &= 1 - \chi_{A^c \cap B^c} \\ &= 1 - \chi_{A^c} \chi_{B^c} \\ &= 1 - (1 - \chi_A)(1 - \chi_B) \\ &= 1 - (1 - \chi_A - \chi_B + \chi_A \chi_B) \\ &= 1 - 1 + \chi_A + \chi_B - \chi_A \chi_B \\ &= \chi_A + \chi_B - \chi_A \chi_B. \end{aligned}$$

Thus $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$. (**4 points**)

Note that $\chi_{A-B} = \chi_{A \cap B^c} = \chi_A \chi_{B^c} = \chi_A(1 - \chi_B)$ so $\chi_{A-B} = \chi_A(1 - \chi_B)$. (**4 points**)

3. (**20 points total**) $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions.

a) Suppose that f, g are injective and let $a, a' \in A$ satisfy $(g \circ f)(a) = (g \circ f)(a')$. Then $g(f(a)) = g(f(a'))$ by definition of function composition. Since g is injective $f(a) = f(a')$. Since f is injective $a = a'$. We have shown that $g \circ f$ is injective. (**6 points**)

b) Suppose that f, g are surjective and let $c \in C$. Since g is surjective $c = g(b)$ for some $b \in B$. Since f is surjective $b = f(a)$ for some $a \in A$. Therefore $(g \circ f)(a) = g(f(a)) = g(b) = c$. We have shown that $g \circ f$ is surjective. (**6 points**)

c) We use Problem 3 of Homework 7 in calculating

$$\begin{aligned}(f^{-1}\circ g^{-1})\circ(g\circ f) &= f^{-1}\circ(g^{-1}\circ(g\circ f)) \\ &= f^{-1}\circ((g^{-1}\circ g)\circ f) \\ &= f^{-1}\circ(I_B\circ f) \\ &= f^{-1}\circ f \\ &= I_A\end{aligned}$$

and

$$\begin{aligned}(g\circ f)\circ(f^{-1}\circ g^{-1}) &= g\circ(f\circ(f^{-1}\circ g^{-1})) \\ &= g\circ((f\circ f^{-1})\circ g^{-1}) \\ &= g\circ(I_B\circ g^{-1}) \\ &= g\circ g^{-1} \\ &= I_C.\end{aligned}$$

(**8 points**)

4. (**20 points total**) We complete the square and find that

$$f(x) = x^2 + 3x - 4 = [(x + \frac{3}{2})^2 - \frac{9}{4}] - 4 = (x + \frac{3}{2})^2 - \frac{25}{4} \geq -\frac{25}{4}.$$

a) For example $f(-1) = f(-2) = -6$; therefore f is *not* injective. (**5 points**)

b) Since $f(x) \geq -\frac{25}{4}$ for all $x \in \mathbf{R}$, f is *not* surjective. Specifically: $c = -7 < -\frac{25}{4}$. Then $f(x) \neq c$ for all $x \in \mathbf{R}$. (**5 points**)

c) By definition

$$\begin{aligned}f^{-1}(\{0, 75/4\}) &= \{x \in \mathbf{R} \,|\, f(x) \in \{0, 75/4\}\} \\ &= \{x \in \mathbf{R} \,|\, f(x) = 0 \;\text{ or }\; f(x) = 75/4\}.\end{aligned}$$

Since $f(x) = 0$ if and only if $f(x) = (x + \frac{3}{2})^2 - \frac{25}{4} = 0$ if and only if $(x + \frac{3}{2})^2 = \frac{25}{4}$ if and only if $x + \frac{3}{2} = \pm\frac{5}{2}$ if and only if $x = 1$ or $x = -4$, and since likewise

$f(x) = \frac{75}{4}$ if and only if $x = \frac{7}{2}$ or $x = -\frac{13}{2}$, it follows that

$$f^{-1}(\{0, 75/4\}) = \{-4, 1, \frac{7}{2}, -\frac{13}{2}\}. \text{ } (\mathbf{5 \ points})$$

d) It is not difficult to show that $f(x)$ in an increasing function on the interval $[0, 1]$. Since $f([0, 1])$ is an interval, $f([0, 1]) = [f(0), f(1)] = [-4, 0]$. (**5 points**)