

# Hour Exam II

(and solution)

Name (print) \_\_\_\_\_

(1) There are *four problems* on this exam. Problem 4 is on the back. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

1. (30 pts.) Let  $A$  and  $B$  be sets.

a) Define the sets  $A \cap B$  and  $A \cup B$  using conditional definitions.

*Solution:*  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$  and  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$  (each 4 points).

b) What does it mean for  $x \notin A \cap B$ ? For  $x \notin A \cup B$ ?

*Solution:*  $x \notin A \cap B$  if and only if  $x \notin A$  or  $x \notin B$  (4 points).  $x \notin A \cup B$  if and only if  $x \notin A$  and  $x \notin B$  (3 points). Part b) is really De Morgan's Laws.

c) Let  $U$  be a universal set and  $A, B \subseteq U$ . Show that  $A = (A \cap B) \cup (A \cap B^c)$  and that the union is disjoint.

*Solution:* Let  $a \in A$ . Then  $a \in U$  since  $A \subseteq U$ . Now  $a \in B$ , in which case  $a \in A \cap B$ , or  $a \notin B$ , in which case  $a \in A \cap (U - B) = A \cap B^c$ . Therefore  $a \in (A \cap B) \cup (A \cap B^c)$  (3 points).

Conversely, suppose  $a \in (A \cap B) \cup (A \cap B^c)$ . Then  $a \in A \cap B$ , in which case  $a \in A$ , or  $a \in A \cap B^c$ , in which case again  $a \in A$ . Thus  $a \in A$  in any event (3 points).

We have established  $A = (A \cap B) \cup (A \cap B^c)$ . To show that the union is disjoint, suppose  $a \in (A \cap B) \cap (A \cap B^c)$ . Then  $a \in B \cap B^c = \emptyset$ . Therefore the intersection  $(A \cap B) \cap (A \cap B^c)$  is the empty set; that is the union  $(A \cap B) \cup (A \cap B^c)$  is disjoint. (2 points).

d) Suppose that  $A = \{1, 0, \pi\}$ . Compute  $P(A)$ .

*Solution:*  $P(A) = \{\emptyset, \{1\}, \{0\}, \{\pi\}, \{1, 0\}, \{1, \pi\}, \{0, \pi\}, \{1, 0, \pi\}\}$  (7 points).

2. (20 pts.) For sets  $A_1, \dots, A_n$  we define the union  $A_1 \cup \dots \cup A_n$  and intersection  $A_1 \cap \dots \cap A_n$  inductively by

$$A_1 \cup \dots \cup A_n = \begin{cases} A_1 & : n = 1; \\ (A_1 \cup \dots \cup A_{n-1}) \cup A_n & : n > 1 \end{cases}$$

and

$$A_1 \cap \cdots \cap A_n = \begin{cases} A_1 & : n = 1; \\ (A_1 \cap \cdots \cap A_{n-1}) \cap A_n & : n > 1 \end{cases}.$$

Now suppose that  $U$  is a universal set and  $A_1, \dots, A_n \subseteq U$ . Use De Morgan's Laws and the definitions above to construct a proof by induction that

$$(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c.$$

[Comment: You may assume  $A_1 \cup \cdots \cup A_m, A_1 \cap \cdots \cap A_m \subseteq U$  for all  $A_1, \dots, A_m \subseteq U$ . The steps of your proof must at least be implicitly justified.]

*Solution:* We wish to prove the following assertion by induction on  $n \geq 1$ : Suppose that  $A_1, \dots, A_n \subseteq U$ . Then  $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$ .

Suppose that  $n = 1$ . Then the assertion boils down to  $A_1^c = A_1^c$  which is true. Therefore the assertion is true for  $n = 1$ . (**6 points**).

Suppose  $n \geq 11$  and the assertion is true for  $n$ . Let  $A_1, \dots, A_{n+1} \subseteq U$ . Then  $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$  (**4 points**). Thus

$$\begin{aligned} (A_1 \cup \cdots \cup A_{n+1})^c &= ((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c \\ &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_{n+1}^c \end{aligned}$$

(**10 points**); the first and last equations follow by our definitions of finite union and intersection, the second equation follows by one of De Morgan's Laws, and the third follows by the induction hypothesis.

3. (25 pts.) Let  $f : A \rightarrow B$  be a function and suppose  $X, Y \subseteq A$ .

a) Show that  $\vec{f}(X \cap Y) \subseteq \vec{f}(X) \cap \vec{f}(Y)$ . [Comment:  $\vec{f}(X) = \{f(x) \mid x \in X\} = f(X)$ , the latter notation was used in class.]

*Solution:* Let  $z \in \vec{f}(X \cap Y)$ . Then  $z = f(x)$  for some  $x \in X \cap Y$ . Since  $x \in X$ ,  $f(x) \in \vec{f}(X)$ . Since  $x \in Y$ ,  $f(x) \in \vec{f}(Y)$ . Therefore  $z = f(x) \in \vec{f}(X) \cap \vec{f}(Y)$  (**6 points**).

b) Suppose that  $f$  is an injection. Show that  $\vec{f}(X \cap Y) = \vec{f}(X) \cap \vec{f}(Y)$ .

*Solution:* In light of part a), we need only show that if  $z \in \vec{f}(X) \cap \vec{f}(Y)$  then  $z \in \vec{f}(X \cap Y)$ . Suppose that  $z \in \vec{f}(X) \cap \vec{f}(Y)$ . Since  $z \in \vec{f}(X)$  there is an  $x \in X$  such that  $z = f(x)$ . Since  $z \in \vec{f}(Y)$  there is a  $y \in Y$  such that  $z = f(y)$ . Thus  $f(x) = z = f(y)$ . Since  $f$  is an injection  $x = y$ . Therefore  $x = y \in X \cap Y$  which means that  $z = f(x) \in \vec{f}(X \cap Y)$  (**5 points**).

- c) Now suppose that  $A = \{1, 2, 3, 4\}$ ,  $B = \{5, 6, 7\}$ ,  $X = \{1, 2\}$ , and  $Y = \{2, 3\}$ . Find a surjection  $g : A \rightarrow B$  such that  $\vec{g}(X \cap Y) \subset \vec{g}(X) \cap \vec{g}(Y)$ . *Justify your answer.*

*Solution:*  $\frac{x}{g(x)} \left| \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 5 & 7 \end{array} \right.$  for example (**4 points**). Since all of the elements of  $B = \{5, 6, 7\}$  are listed as outputs  $g$  is a surjection (**4 points**). Note that

$$\vec{g}(X) = \vec{g}(\{1, 2\}) = \{g(1), g(2)\} = \{5, 6\} = \{g(3), g(2)\} = \vec{g}(\{3, 2\}) = \vec{g}(Y)$$

which means that  $\vec{g}(X) \cap \vec{g}(Y) = \{5, 6\}$  (**4 points**). Since

$$\vec{g}(X \cap Y) = \vec{g}(\{2\}) = \{6\}$$

it follows that  $\vec{g}(X \cap Y) \subset \vec{g}(X) \cap \vec{g}(Y)$  (**2 points**).

4. (25 pts.) A committee of 7 is to be formed from a group of 10 people. At least one of two individuals  $A$  and  $B$  is to be on the committee. Let  $X$  be the set of those committees of 7 which include  $A$  and let  $Y$  be the set of those committees of 7 which include  $B$ .

- a) Determine  $|X|$  and  $|Y|$  explicitly.

*Solution:*  $|X| = |Y| = \binom{10-1}{7-1} = \binom{9}{6} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84$  (**6 points**).

- b) Determine explicitly the number of the committees of 7 which include both  $A$  and  $B$ . Express the set of these committees in terms of  $X$  and  $Y$ .

*Solution:* The set in question is  $X \cap Y$  (**3 points**) which has

$$\binom{10-2}{7-2} = \binom{8}{5} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$$

elements (**4 points**).

- c) Use the inclusion-exclusion principle to determine explicitly the number of committees of 7 which include either  $A$  or  $B$  (perhaps both). Express the set of these committees in terms of  $X$  and  $Y$ .

*Solution:* The set in question is  $X \cup Y$  (**3 points**) and by the inclusion-exclusion principle

$$|X \cup Y| = |X| + |Y| - |X \cap Y| = 84 + 84 - 56 = 112$$

elements. Thus there are 112 committees which include  $A$  or  $B$  (**9 points**).