Remarks on §10.3 and §10.4

by David Radford, November 24, 2003

Throughout $\boldsymbol{R}$ is the set of real numbers and all functions $f(x)$ are *integrable.*

1 Fourier series and orthogonality relations

Here we discuss some of the basic ideas presented in §10.3. We have tried to simplify the mathematics.

Let $T > 0$ and let $f : [-T, T] \longrightarrow \boldsymbol{R}$ be a function. Recall that

$$\int_{-T}^{T} f(x)\,dx = 0 \qquad \text{if } f \text{ is an } \textit{odd} \text{ function.} \tag{1}$$

As a consequence

$$\int_{-T}^{T} \cos\frac{n\pi}{T}x \sin\frac{m\pi}{T}x\,dx = 0 \tag{2}$$

for all integers n, m. We will show that

$$\int_{-T}^{T} \cos\frac{n\pi}{T}x \cos\frac{m\pi}{T}x\,dx = \int_{-T}^{T} \sin\frac{n\pi}{T}x \sin\frac{m\pi}{T}x\,dx = 0 \tag{3}$$

for all distinct *non-negative* integers n, m and

$$\int_{-T}^{T} \cos\frac{n\pi}{T}x \cos\frac{n\pi}{T}x\,dx = \int_{-T}^{T} \sin\frac{n\pi}{T}x \sin\frac{n\pi}{T}x\,dx = T \tag{4}$$

for all positive integers n.

To prove (3) and (4) we recall that

$$\cos(a - b) = \cos a \cos b + \sin a \sin b,$$

and

$$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

for all $a, b \in \boldsymbol{R}$. The second equation follows from the first since $a + b = a - (-b)$, cosine is an even function, and sine is an odd function. From these two equation we derive

$$\cos a \cos b = \frac{1}{2}(\cos(a + b) + \cos(a - b))$$

and

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

In particular

$$\cos \frac{n\pi}{T}x \cos \frac{m\pi}{T}x = \frac{1}{2}\left(\cos \frac{(n+m)\pi}{T}x + \cos \frac{(n-m)\pi}{T}x\right) \tag{5}$$

and

$$\sin \frac{n\pi}{T}x \sin \frac{m\pi}{T}x = \frac{1}{2}\left(\cos \frac{(n-m)\pi}{T}x - \cos \frac{(n+m)\pi}{T}x\right) \tag{6}$$

for all integers n, m. Let k be *any* non-zero integer. Since

$$\int_{-T}^{T} \cos \frac{k\pi}{T}x \, dx = \frac{T}{k\pi}\left(\sin \frac{k\pi}{T}T - \sin \frac{k\pi}{T}(-T)\right) = 0$$

and $\cos \frac{0\pi}{T}x = 1$ it is easy to see that (3) and (4) follow from (5) and (6).

To continue, we define a "dot product" for all integrable functions $f, g : [-T, T] \longrightarrow \boldsymbol{R}$ by

$$<f, g> = \int_{-T}^{T} f(x)g(x)\, dx.$$

This is meant to be reminiscent of the dot product on $\boldsymbol{R}^n$. Note that

$$<f, f> = \int_{-T}^{T} f(x)^2 \, dx \geq 0.$$

Thus, as in the case of $\boldsymbol{R}^n$, we can define length by

$$||f|| = \sqrt{<, f, f>} = \left(\int_{-T}^{T} f(x)^2 \, dx\right)^{1/2}.$$

Observe that $||f||^2 = <f, f>$.

Recall that two vectors in $\boldsymbol{R}^n$ are orthogonal, or are at right angles to one another, if their dot product is zero. The equation of (2) is equivalent to

$$<\cos \frac{n\pi}{T}x, \sin \frac{m\pi}{T}x> = 0 \tag{7}$$

for all integers n, m. Observe that (3) can be reformulated

$$<\cos\frac{n\pi}{T}x, \cos\frac{m\pi}{T}x> \; = \; <\sin\frac{n\pi}{T}x, \sin\frac{m\pi}{T}x> = 0 \tag{8}$$

for all distinct non-negative integers m, n and (4) can be expressed as

$$||\cos\frac{n\pi}{T}x||^2 = ||\sin\frac{n\pi}{T}x||^2 = T \tag{9}$$

for all positive integers n. The equations of (7) and (8) are *orthogonality relations*. By (9) the functions $\cos\frac{n\pi}{T}x$ and $\sin\frac{n\pi}{T}x$, where n is a positive integer, all have length $\sqrt{T}$. When $n = 0$ observe that

$$||\cos\frac{0\pi}{T}x||^2 = ||1||^2 = \int_{-T}^{T} 1\,dx = 2T \tag{10}$$

and

$$||\sin\frac{0\pi}{T}x||^2 = ||0||^2 = \int_{-T}^{T} 0\,dx = 0.$$

The dot product we have just defined has the algebraic properties of the dot product of $\boldsymbol{R}^n$. We now want to apply the ideas developed above to Fourier series. We want to *motivate* the Euler equations.

Suppose that the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{T}x + b_n\sin\frac{n\pi}{T}x\right)$$

is meaningful, where the coefficients are real numbers. Let m be a positive integer. Then, treating infinite sums as finite sums, we calculate

$$\begin{aligned}
&<f(x), \sin\frac{m\pi}{T}x> \\
&= \; <\frac{a_0}{2},\ \sin\frac{m\pi}{T}x> + \sum_{n=1}^{\infty}\left(<a_n\cos\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x> + <b_n\sin\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x>\right) \\
&= \; \frac{a_0}{2}<1, \sin\frac{m\pi}{T}x> + \sum_{n=1}^{\infty}\left(a_n<\cos\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x> + b_n<\sin\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x>\right) \\
&= \; \frac{a_0}{2}0 + \sum_{n=1}^{\infty}\left(a_n<\cos\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x> + b_n<\sin\frac{n\pi}{T}x,\ \sin\frac{m\pi}{T}x>\right)
\end{aligned}$$

$$\begin{aligned} &= b_m \langle \sin \frac{m\pi}{T}x,\ \sin \frac{m\pi}{T}x \rangle \\ &= b_m || \sin \frac{m\pi}{T}x ||^2 \\ &= b_m T. \end{aligned}$$

The third equation follows by (1) since $\sin \frac{m\pi}{T}x$ is an odd function, the fourth follows by the orthogonality relations (7) and (8), and the fifth follows by (9). Our calculation results in

$$b_n = \frac{1}{T}\langle f(x), \sin \frac{n\pi}{T}x \rangle = \frac{1}{T}\int_{-T}^{T} f(x) \sin \frac{n\pi}{T}x \, dx$$

for all *positive* integers n. Thus we are led to *define*

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x) \sin \frac{n\pi}{T}x \, dx \tag{11}$$

for all *positive* integers n.

By virtue of the orthogonality relations and (9) the calculation above holds when $\sin \frac{m\pi}{T}x$ is replaced by $\cos \frac{m\pi}{T}x$. We obtain from it

$$a_n = \frac{1}{T}\langle f(x), \cos \frac{n\pi}{T}x \rangle = \frac{1}{T}\int_{-T}^{T} f(x) \cos \frac{n\pi}{T}x \, dx$$

for positive integers n. Using the orthogonality relations and (10) the calculation above yields

$$a_0 = \frac{1}{T}\langle f(x), \cos \frac{0\pi}{T}x \rangle = \frac{1}{T}\int_{-T}^{T} f(x) \cos \frac{0\pi}{T}x \, dx = \frac{1}{T}\int_{-T}^{T} f(x) \, dx.$$

when $\sin \frac{m\pi}{T}x$ is replaced by $\cos \frac{0\pi}{T}x = 1$.

Thus we are led to *define*

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x) \cos \frac{n\pi}{T}x \, dx \tag{12}$$

or all *non-negative* integers n. The equations of (11) and (12) are referred to as the *Euler equations.*

Generally, if $f : [-T, T] \longrightarrow \boldsymbol{R}$ is integrable, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{T}x + b_n \sin \frac{n\pi}{T}x \right),$$

where the series on the right is referred to as the *Fourier series of* $f(x)$, where the coefficients are defined by the Euler equations.

2 Even-odd decomposition of certain functions and the resulting Fourier decompositions

This section and the next deal with the basic ideas of §10.4.

Let $T > 0$ and suppose that $f : [-T, T] \longrightarrow \boldsymbol{R}$ is a function. We define functions $f_e, f_o : [-T, T] \longrightarrow \boldsymbol{R}$ by

$$f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

and

$$f_e(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

for all $-T \leq x \leq T$. Then f_e is an even function, f_o is an odd function, and

$$f = f_e + f_o$$

which expresses f as the sum of an even function and an odd function. *This decomposition is unique.* For suppose that $f = \mathsf{f_e} + \mathsf{f_o}$, where $\mathsf{f_e}, \mathsf{f_o} : [-T, T] \longrightarrow \boldsymbol{R}$ is even (respectively odd). From the equation $f_e + f_o = f = \mathsf{f_e} + \mathsf{f_o}$ we derive the equation $f_e - \mathsf{f_e} = \mathsf{f_o} - f_o$. The left hand side of the last equation is an even function and the right hand side is an odd function. Since the only function which is both even and odd is the zero function, $f_e - \mathsf{f_e} = 0 = \mathsf{f_o} - f_o$, or $f_e = \mathsf{f_e}$ and $\mathsf{f_o} = f_o$.

We now relate the Fourier series of $f(x)$ to those of $f_e(x)$ and $f_o(x)$. First of all recall by (1) that if $g(x) : [-T, T] \longrightarrow \boldsymbol{R}$ is an *odd* function then

$$\int_{-T}^{T} g(x)\, dx = 0.$$

Suppose that

$$f(x) \quad \sim \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{T}x + b_n \sin\frac{n\pi}{T}x\right).$$

Since $f_o(x)\cos\frac{n\pi}{T}x$ and $f_e(x)\sin\frac{n\pi}{T}x$ are odd functions we have

$$a_n \quad = \quad \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x\, dx$$

$$
\begin{aligned}
&= \frac{1}{T}\int_{-T}^{T} f_e(x)\cos\frac{n\pi}{T}x\,dx + \frac{1}{T}\int_{-T}^{T} f_o(x)\cos\frac{n\pi}{T}x\,dx \\
&= \frac{1}{T}\int_{-T}^{T} f_e(x)\cos\frac{n\pi}{T}x\,dx \qquad + \qquad 0
\end{aligned}
$$

for all non-negative integers n and

$$
\begin{aligned}
b_n &= \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x\,dx \\
&= \frac{1}{T}\int_{-T}^{T} f_e(x)\sin\frac{n\pi}{T}x\,dx + \frac{1}{T}\int_{-T}^{T} f_o(x)\sin\frac{n\pi}{T}x\,dx \\
&= 0 \qquad + \qquad \frac{1}{T}\int_{-T}^{T} f_o(x)\sin\frac{n\pi}{T}x\,dx
\end{aligned}
$$

for all positive integers n. Therefore

$$
f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{T}x\,dx \tag{13}
$$

and

$$
f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{T}x\,dx, \tag{14}
$$

$$
a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x\,dx = \frac{1}{T}\int_{-T}^{T} f_e(x)\cos\frac{n\pi}{T}x\,dx
$$

for all $n \geq 0$, and

$$
b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x\,dx = \frac{1}{T}\int_{-T}^{T} f_o(x)\sin\frac{n\pi}{T}x\,dx
$$

for all $n \geq 1$.

3 Extension to even or odd functions in certain cases and resulting Fourier series

Let $T > 0$ and suppose that $f : [0, T] \longrightarrow \boldsymbol{R}$ is an integrable function. There is a natural way of extending $f(x)$ to an even function $F_e(x)$ and an odd function $F_o(x)$, where $F : [-T, T] \longrightarrow \boldsymbol{R}$ is integrable. The *Fourier sine series of* $f(x)$ *on* $[0, T]$ is the Fourier series of $F_o(x)$, where $0 \leq x \leq T$, and the *Fourier cosine series of* $f(x)$ *on* $[0, T]$ is the Fourier series of $F_e(x)$, where $0 \leq x \leq T$.

Define $F : [-T, T] \longrightarrow \boldsymbol{R}$ by

$$F(x) = \begin{cases} 0 & : \quad -T \leq x < 0 \\ 2f(x) & : \quad 0 \leq x \leq T \end{cases}$$

The “2” in this definition will explain where “2” comes in the formulas of §10.4.

We first note that

$$F_e(x) = f(x) = F_o(x)$$

for all $0 \leq x \leq T$. To establish this suppose $0 < x \leq T$. Then

$$F_e(x) = \frac{1}{2}\left(F(x) + F(-x)\right) = \frac{1}{2}(2f(x) + 0) = f(x).$$

Observe that

$$F_e(0) = \frac{1}{2}\left(F(0) + F(-0)\right) = \frac{1}{2}(2f(0)) = f(0).$$

We have shown $F_e(x) = f(x)$ for all $0 \leq x \leq T$. Thus for such an x the equation

$$2f(x) = F(x) = F_e(x) + F_o(x) = f(x) + F_o(x)$$

follows which implies $F_o(x) = f(x)$ also.

Let

$$F(x) \quad \sim \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{T} x + b_n \sin \frac{n\pi}{T} x \right).$$

By (13) and (14) we have

$$F_e(x) \quad \sim \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{T} x.$$

and

$$F_o(x) \quad \sim \quad \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T} x.$$

By definition we have

$$a_n = \frac{1}{T}\int_{-T}^{T} F(x)\cos\frac{n\pi}{T}x\,dx = \frac{1}{T}\int_{0}^{T} 2f(x)\cos\frac{n\pi}{T}x\,dx,$$

so

$$a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi}{T}x\,dx$$

for $n \geq 0$, and

$$b_n = \frac{1}{T}\int_{-T}^{T} F(x)\sin\frac{n\pi}{T}x\,dx = \frac{1}{T}\int_{0}^{T} 2f(x)\sin\frac{n\pi}{T}x\,dx,$$

so

$$b_n = \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi}{T}x\,dx$$

for $n \geq 1$.