Remarks regarding Hour Exam II

Radford's Section (67152)

November 13, 2003

1. My apologies to the TU 10 Discussion Section for forgetting to return Quiz 5 in class yesterday. They are in an envelope on my door. *Your TA was (is to) to return Quiz 5 to the other sections.*

2. Solutions to Quizzes 4 and 5 are posted on our math 220 home page.

3. The homework which was not collected Wednesday will be collected Monday.

Hour Exam II

(1) *Date* and *time*: Friday, November 14, 12–12:50PM.

(2) *Location*: TUESDAY discussion sections, LEC C3; THURSDAY discussion sections, LEC C4.

(3) *Syllabus*: According to the course home page (exams section): 5.1, 5.3; 7.1–7.6; 8.1, 8.3.

(4) *Format*: 5 questions, partial credit.

(5) *ID Check*: Bring a photo ID. There will be an ID check.

(6) *Tables*: You will be given the following tables on the back of your exam.

Table of Laplace transforms

$f(t)$	$\mathcal{L}(f)(s)$	
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$n \geq 0, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$	$n \geq 0, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$	$n \geq 0, \quad s > a$

Some General Properties of the Laplace Transform

$$\mathcal{L}(e^{at}f(t))(s) = \mathcal{L}(f)(s-a)$$

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$$

$$\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$$

$$\mathcal{L}(f(t-a)u(t-a))(s) = e^{-as}\mathcal{L}(f)(s)$$

Here are highlights of our review Monday and Wednesday of this week with some further thoughts.

1. How would the system of equations for Problem 31, page 285, be different if the second tank had 75 liters instead of 100? See Problem 6 of the sample exam dated November 15, 2002.

2. Concerning connected tanks, consider the system of equations

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned}$$

where $x = x(t), y = y(t)$, and a, b, c, d are real numbers. By the method of elimination we derived the equations

$$(D^2 - (a+d)D + (ad - bc)I)(x) = 0$$

$$(D^2 - (a+d)D + (ad - bc)I)(y) = 0.$$

Thus x and y have the same general form. *It is important to take x and y which solve these equations and plug them back into the original system of equations.* For example, solving the latter equations, given the system

$$\begin{aligned} x' &= x - 4y \\ y' &= x + y, \end{aligned}$$

yields

$$x = e^t(c_1 \cos 2t + c_2 \sin 2t)$$

and

$$y = e^t(c_3 \cos 2t + c_4 \sin 2t).$$

Since

$$x' = (e^t)'(c_1 \cos 2t + c_2 \sin 2t) + e^t(c_1 \cos 2t + c_2 \sin 2t)' = x + e^t(-2c_1 \sin 2t + 2c_2 \cos 2t)$$

the equation $x' = x - 4y$ becomes

$$e^t(-2c_1 \sin 2t + 2c_2 \cos 2t) = -4e^t(c_3 \cos 2t + c_4 \sin 2t).$$

Therefore

$$c_3 = -\frac{1}{2}c_2 \qquad \text{and} \qquad c_4 = \frac{1}{2}c_1.$$

Check that

$$x = e^t(c_1 \cos 2t + c_2 \sin 2t) \qquad \text{and} \qquad y = e^t(-(c_2/2)\cos 2t + (c_1/2)\sin 2t)$$

do indeed solve the system (This is the general solution to the system).

For another example, consider

$$\begin{aligned} x' &= 4x + y \\ y' &= -2x + y. \end{aligned}$$

from which we deduce

$$x = c_1 e^{2t} + c_2 e^{3t} \qquad \text{and} \qquad y = c_3 e^{2t} + c_4 e^{3t}.$$

Since

$$2c_1 e^{2t} + 3c_2 e^{3t} = x' = 4x + y = (4c_1 + c_3)e^{2t} + (4c_2 + c_4)e^{3t}$$

we have the system

$$\begin{aligned} 2c_1 &= 4c_1 + c_3 \\ 3c_2 &= 4c_2 + c_4. \end{aligned}$$

Therefore $c_3 = -2c_1$ and $c_4 = -c_2$. Check that

$$x = c_1 e^{2t} + c_2 e^{3t} \qquad \text{and} \qquad y = -2c_1 e^{2t} - c_2 e^{3t}$$

are indeed solutions to the system (This is the general solution to the system).

3. Taylor polynomials. Let $y = f(x)$ and $x_0 = 0$. Then

$$p_n(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n$$

for all $n \geq 0$. For the following, see the solution to problem 1 on Quiz 5.

$$y^{(2)} = (x+4)^3 + y^2, \qquad y(0) = 2, \;\; y'(0) = -3.$$

We calculate $p_n(x)$ for $0 \leq n \leq 4$.

$$\begin{aligned} y^{(3)} &= 3(x+4)^2 + 2yy^{(1)} \\ y^{(4)} &= 6(x+4) + 2(y^{(1)}y^{(1)} + yy^{(2)}) \end{aligned}$$

so

$$\begin{array}{rcllcl} y^{(2)}(0) & = & (0+4)^3+2^2 & & = & 68, \\ y^{(3)}(0) & = & 3(0+4)^2+2(2)(-3) & & = & 36, \\ y^{(4)}(0) & = & 6(0+4)+2((-3)^2+(2)(68)) & & = & 314. \end{array}$$

Therefore

$$\begin{aligned} p_0(x) &= 2, \\ p_1(x) &= 2-3x, \\ p_2(x) &= 2-3x+34x^2, \\ p_3(x) &= 2-3x+34x^2+6x^3, \\ p_4(x) &= 2-3x+34x^2+6x^3+\frac{157}{12}x^4. \end{aligned}$$

The first *five* non-zero terms of the Taylor series expansion of the solution to the given equation with initial conditions are given by $2-3x+34x^2+6x^3+\frac{157}{12}x^4+\cdots$.

The Taylor series expansion of $y=f(x)$ about $x_0=0$ is given by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n.$$

4. Series solutions. Let $y=f(x)$, $x_0=0$, and $y=\sum_{n=0}^{\infty}a_nx^x$ be a series expansion of $y=f(x)$. Then

$$y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad \text{and} \qquad y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Some summation gymnastics.

$$\begin{aligned} (x^2+4)y &= \sum_{n=0}^{\infty}a_nx^{n+2}+\sum_{n=0}^{\infty}4a_nx^n \\ &= \sum_{m=2}^{\infty}a_{m-2}x^m+\sum_{n=0}^{\infty}4a_nx^n \\ &= \sum_{n=2}^{\infty}a_{n-2}x^n+\sum_{n=0}^{\infty}4a_nx^n \\ &= \sum_{n=2}^{\infty}(a_{n-2}+4a_n)x^n+4a_0+4a_1x \\ &= 4a_0+4a_1x+\sum_{n=2}^{\infty}(a_{n-2}+4a_n)x^n \end{aligned}$$

and

$$\begin{aligned} x^2 + 4y &= x^2 + \sum_{n=0}^{\infty} 4a_n x^n \\ &= 4a_0 + 4a_1 x + (4a_2 + 1)x^2 + \sum_{n=3}^{\infty} 4a_n x^n. \end{aligned}$$

5. The equation

$$y'' + ay' + by = g(t)$$

is the one to which the Method of Laplace Transforms is most commonly applied in the text. Let

$$Y(s) = (\mathcal{L}(y))(s) \qquad \text{and} \qquad G(s) = \mathcal{L}(g)(s).$$

From the second table above we deduce

$$Y(s) = \frac{G(s) + (s+a)y(0) + y'(0)}{s^2 + as + b}.$$

Thus if $G(s) = 0$, or more generally is a rational function in s, the right hand side of the equation above is a rational function in s and the method of partial fractions applies for the solution of

$$y = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{G(s) + (s+a)y(0) + y'(0)}{s^2 + as + b}\right).$$