MATH 220 Hour Exam I SOLUTION **Radford** **10/16/03**

1. (20 pts.) Find the solution to $\dfrac{dy}{dx} = (x + \cos x)(y + 2)$ which satisfies $y(0) = 7$.

   Solution: Separating variables we have $\dfrac{dy}{y+2} = (x + \cos x)\,dx$. (**7 points**) Integrating both sides yields $\ln(y + 2) = \dfrac{x^2}{2} + \sin x + C$ and therefore $y + 2 = e^{\frac{x^2}{2}+\sin x}e^C = Ae^{\frac{x^2}{2}+\sin x}$ for some constant A (**8 points**). Now $9 = 7 + 2 = y(0) + 2 = Ae^0 = A$ so $y = 9e^{\frac{x^2}{2}+\sin x} - 2$. (**5 points**)

2. (20 pts.) Find the general solution

   a) to $y'' + y' + 3y = 0$;

   Solution: The auxiliary equation is $r^2+r+3 = 0$ which has roots $r = -\dfrac{1}{2} \pm \dfrac{\sqrt{11}}{2}\imath$. (**4 points**) Thus the general solution is $y = c_1e^{-(1/2)x}\cos(\dfrac{\sqrt{11}}{2}x) + c_2e^{-(1/2)x}\sin(\dfrac{\sqrt{11}}{2}x)$ (**6 points**)

   b) to $y'' + 14y' + 49y = 0$.

   Solution: The auxiliary equation is $r^2+14r+49 = (r+7)^2 = 0$ which has one real root $r = -7$. (**3 points**) Thus the general solution is $y = c_1e^{-7x} + c_2xe^{-7x}$. (**7 points**)

3. (20 pts.) Consider the equation $\dfrac{dy}{dx} = e^{5x} - 3y$.

   a) Given that $y(0) = \dfrac{7}{8}$, use Euler's method to *approximate* $y(\dfrac{1}{2})$ in one step (thus $h = \dfrac{1}{2}$).

   Solution: $(\dfrac{1}{2}) \approx y(0) + hf(0, y(0))$ (**3 points**) $= \dfrac{7}{8} + (\dfrac{1}{2})(e^0 - 3(\dfrac{7}{8})) = \dfrac{7}{8} + \dfrac{1}{2} - \dfrac{21}{16} = \dfrac{1}{16}$. (**4 points**)

   b) Find the general solution to $\dfrac{dy}{dx} + 2y = e^{5x}$.

   Solution: $\mu = e^{\int 2\,dx} = e^{2x}$. (**6 points**) Therefore

$$y = \frac{1}{\mu}\left(\int \mu e^{5x}\,dx + C\right) = \frac{1}{e^{2x}}\left(\int e^{2x+5x}\,dx + C\right) = e^{-2x}\left(\frac{1}{7}e^{7x} + C\right) = \frac{e^{5x}}{7} + Ce^{-2x}.$$

   (**7 points**)

4. (20 pts.) Consider the equation $y'' - 5y' = 4x - 7$ $(*)$.

a) Find a fundamental set of solutions to $y'' - 5y' = 0$.

Solution: The auxiliary equation is $r^2 - 5r = 0$ which has roots $r = 0, 5$. Thus $\{e^{0x}, e^{5x}\} = \{1, e^{5x}\}$ is a fundamental set of solutions. (**4 points**)

b) Compute the Wronskian of your solutions of part a).

Solution: $W(1, e^{5x}) = \begin{vmatrix} 1 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} = (1)(5e^{5x}) - (e^{5x})(0) = 5e^{5x}$. (**4 points**)

c) Find a particular solution to (*).

Solution: $y_p = x(Ax + B)$ since B is a solution to the homogeneous equation $y'' - 5y' = 0$. Since $y_p' = 2Ax + B$ and $y_p' = 2A$ we have $2A - 5(2AX + B) = y_p'' - 5y_p' = 4x - 7$ which is equivalent to the linear system

$$\begin{aligned} 2A - 5B &= -7 \\ -10A &= 4 \end{aligned}$$

Therefore $A = -\frac{2}{5}, B = \frac{31}{25}$ which means that $y_p = -\frac{2}{5}x^2 + \frac{31}{25}x$ is a particular solution. (**7 points**)

d) Find the general solution to (*).

Solution: $y = y_h + y_p = c_1 1 + c_2 e^{5x} + -\frac{2}{5}x^2 + \frac{31}{25}x$. (**5 points**)

5. (20 pts.) A brine solution is flowing into a tank containing 40 gallons of solution at the rate of 8 gallons per minute and is flowing out of the tank at the same rate. Suppose that the tank is well stirred and the solution flowing in contains 25% salt. Given that the tank contained 22 gallons of salt initially, find:

a) the amount of salt in the tank at time t;

Solution: Let $x = x(t)$ be the amount of salt in the tank at time t. Then $\frac{dx}{dt} = \textit{rate in} - \textit{rate out}$ so $\frac{dx}{dt} = (.25)8 - (\frac{x}{40})8 = 2 - \frac{x}{5}$ (**5 points**) Solving we have $x = 10 + Ce^{-t/5}$. (**5 points**) Since $x(0) = 22$ it follows that $C = 12$ and thus $x = 10 + 12e^{-t/5}$. (**5 points**)

b) the amount of salt in the tank after ten minutes. (You may leave your answer as a numerical expression.)

Solution: $x(10) = 10 + 12e^{-2}$. (**5 points**)