MATH 220 Hour Exam II Solution **Radford** **11/24/03**

1. (20 pts.) Find the *general solution* to the system of equations

$$\begin{aligned} x' &= 2x - 9y \\ y' &= x + 2y. \end{aligned}$$

Solution: By the elimination method we have

$$\begin{aligned} (D^2 - 4D + 13I)(x) &= 0 \\ (D^2 - 4D + 13I)(y) &= 0. \end{aligned}$$

By the quadratic formula $r^2 - 4r + 13 = 0$ has roots $r = 2 \pm 3\imath$. Therefore

$$x = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) \;\; (\mathbf{7} \;\; points)$$

and

$$y = e^{2t}(c_3 \cos 2t + c_4 \sin 2t). \;\; (\mathbf{6} \;\; points)$$

Since

$$y' = (e^{2t})'(c_3 \cos 3t + c_4 \sin 3t) + e^{2t}(c_3 \cos 3t + c_4 \sin 3t)' = 2y + e^{2t}(-2c_3 \sin 3t + 2c_4 \cos 3t)$$

the equation $y' = x + 2y$ becomes

$$e^{2t}(c_1 \cos 3t + c_2 \sin 2t) = e^{2t}(-3c_3 \sin 2t + 3c_4 \cos 2t).$$

Therefore

$$c_3 = -\frac{1}{3}c_2 \;\; (\mathbf{3} \;\; points) \qquad \text{and} \qquad c_4 = \frac{1}{3}c_1 \;\; (\mathbf{4} \;\; points).$$

2. (25 pts.) Compute: $(a)\;\; \mathcal{L}(5t^7 + 4\cos 6t - 2e^{3t})$ $\qquad (b)\;\; \mathcal{L}^{-1}\left(\dfrac{s+11}{2s^2+8s+18}\right)$.

Solution:

(a)

$$\begin{aligned} \mathcal{L}(5t^7 + 4\cos 6t - 2e^{3t}) &= 5\mathcal{L}(t^7) + 4\mathcal{L}(\cos 6t) - 2\mathcal{L}(e^{3t}) \\ &= 5\frac{7!}{s^8} + 4\frac{s}{s^2+36} - 2\frac{1}{s-3} \;\; (\mathbf{4} \;\; points\ each\ of\ the\ three\ parts). \end{aligned}$$

(b) Since

$$\frac{s+11}{2s^2+8s+18} = \left(\frac{1}{2}\right)\left(\frac{s+11}{s^2+4s+9}\right) = \left(\frac{1}{2}\right)\left(\frac{s+2}{(s+2)^2+(\sqrt{5})^2} + \frac{9}{(s+2)^2+(\sqrt{5})^2}\right)$$

the answer is

$$\left(\frac{1}{2}\right)\left(e^{-2t}\right)\left(\cos\sqrt{5}t + \frac{9}{\sqrt{5}}\sin\sqrt{5}t\right) \quad (\mathbf{2,3,4,4}\ \textit{points for the respective parts}).$$

3. (20 pts.) Use the method of Laplace transforms to solve the initial value problem

$$y'' - 9y' + 14y = 0, \qquad y(0) = 4,\ \ y'(0) = 31.$$

Show all of your work, including the details of work to solve the partial fraction expansion.

Solution: From the second table appended to the test, or the formula derived in class, we deduce

$$Y(s) = \frac{(s-9)4+31}{s^2-9s+14} = \frac{4s-5}{(s-2)(s-7)} \quad (\mathbf{8}\ \textit{points}).$$

By the method of partial fractions we have

$$\frac{4s-5}{(s-2)(s-7)} = \frac{A}{s-2} + \frac{B}{s-7} = \frac{A(s-7)+B(s-2)}{(s-2)(s-7)}$$

From which the linear system

$$\begin{aligned} A+B &= 4 \\ -7A-2B &= -5, \end{aligned}$$

or

$$\begin{aligned} A+B &= 4 \\ 7A+2B &= 5. \end{aligned}$$

Multiplying the first by 2 and subtracting the resulting equation from the second we have $5A = -3$, or equivalently $A = -\frac{3}{5}$ ($\mathbf{4}$ *points*). Therefore $B = \frac{23}{5}$ ($\mathbf{3}$ *points*). Taking the inverse Laplace transform we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\left(-\frac{3}{5}\right)\left(\frac{1}{s-2}\right) + \left(\frac{23}{5}\right)\left(\frac{1}{s-7}\right)\right) \\ &= \left(-\frac{3}{5}\right)\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \left(\frac{23}{5}\right)\mathcal{L}^{-1}\left(\frac{1}{s-7}\right) \\ &= \left(-\frac{3}{5}\right)e^{2t} + \left(\frac{23}{5}\right)e^{7t} \quad (\mathbf{5}\ \textit{points}) \end{aligned}$$

4. (15 pts.) Find the Laplace transform $Y(s) = \mathcal{L}(y)(s)$ of $y = y(t)$, given that

$$y' + 5y = \sin 3t \qquad \text{and} \qquad y(0) = 4.$$

Find an *explicit* expression for $Y(s)$ as the ratio of polynomials in s; do *not* solve for $y(t)$.

Solution: From the second table appended to the test we derive

$sY(s) - 4 + 5Y(s) = \dfrac{3}{s^2+9}$ (**5, 4** points for the left and right sides of the equation respectively).

Therefore

$$Y(s) = \frac{\frac{3}{s^2+9} + 4}{s+5} = \frac{4s^2+39}{(s+5)(s^2+9)}, \quad or \quad \frac{s+5}{s^3+5s^2+9s+45} \quad (\mathbf{6} \ \textit{points}).$$

5. (20 pts.) Consider the equation $y'' + 3y' + xy = 0$.

a) Suppose that $y = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution of the equation. Find a_2 and a_3 in terms of a_0, a_1.

Solution: Since $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$, $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$, we have

$$3y' = \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n$$

and

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1}x^m = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Thus

$$y'' + 3y' + xy = (2a_0 + 3a_1) + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + a_{n-1})x^n$$

which means

$$2a_2 + 3a_1 = 0$$

and

$$(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + a_{n-1} = 0$$

for all $n \geq 1$. Therefore

$$a_2 = -\frac{3}{2}a_1 \quad (\mathbf{5} \ \textit{points})$$

and $6a_3 + 6a_2 + a_0 = 0$ from which we deduce

$$a_3 = \frac{3}{2}a_1 - \frac{1}{6}a_0. \quad (\mathbf{5} \ \textit{points})$$

b) Determine the Taylor polynomial $p_4(x)$ approximating the solution at $x_0 = 0$ of the initial value problem $y'' + 3y' + xy = 0$, where $y(0) = 1$ and $y'(0) = -2$.

b) We will return to the original equation to compute $p_4(x)$. Since

$$\begin{aligned} y^{(2)} + 3y^{(1)} + xy &= 0, \\ y^{(3)} + 3y^{(2)} + y + xy^{(1)} &= 0, \\ y^{(4)} + 3y^{(3)} + y^{(1)} + y^{(1)} + xy^{(2)} &= 0, \end{aligned}$$

we have

$$\begin{aligned} y^{(2)}(0) &= -3(-2) - (0)(-2) &&= 6, \\ y^{(3)}(0) &= -3(6) - 1 - (0)(-2) &&= -19, \\ y^{(4)}(0) &= -3(-19) - 2(-2) - (0)(6) &&= 61. \end{aligned}$$

Thus

$p_4(x) = (1 - 2x + 3x^2) - (\frac{19}{6}x^3) + (\frac{61}{24}x^4)$ (**4, 4, 2** *points for the parenthesized parts respectively*).