Remarks on §10.2 and §10.5

by David Radford, November 27, 2003

Let $L > 0$ and $u = u(x, t)$ be a function defined for $0 \leq x \leq L$ and $t \geq 0$. The one-dimensional heat flow equation (or heat equation) is

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x, t) \tag{1}$$

where $\beta > 0$ and $P = P(x, t)$ is a function defined for $0 \leq x \leq L$ and $t \geq 0$ as well. An important special case of is the heat equation is

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \tag{2}$$

which occurs when $P = P(x, t) = 0$.

Differentiation, and partial differentiation, are *linear* operators. Since compositions and linear combinations of linear operators are linear operators, the operator

$$\mathcal{L} = \frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2}$$

is linear. See §4.2. Observe that

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \quad \text{is equivalent to} \quad \mathcal{L}(u) = 0.$$

Thus if u_1, u_2 are solutions to (2), the linearity of $\mathcal{L}$ gives

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2) = c_1 0 + c_2 0 = 0$$

and thus the linear combination $c_1 u_1 + c_2 u_2$ is also a solution to (2) for all real numbers c_1, c_2.

Observe that

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x, t) \quad \text{is equivalent to} \quad \mathcal{L}(u) = P,$$

where again $P = P(x, t)$. There is a very close connection between solutions of (2) and those of (1); that is solutions of $\mathcal{L}(u) = 0$ and solutions of $\mathcal{L}(u) =$

P. See the procedure for solving nonhomogeneous equations in terms of homogeneous equations described on pages 201–202.

Suppose that u and u_p are solutions to (1), which is to say $\mathcal{L}(u) = P$ and $\mathcal{L}(u_p) = P$. Then the calculation

$$\mathcal{L}(u - u_p) = \mathcal{L}(u) - \mathcal{L}(u_p) = P - P = 0$$

shows that $u_h = u - u_p$ is a solution to (2). Thus

$$u = u_p + u_h,$$

where u_p is a *particular* solution to (1) and u_h is some solution to (2). It is easy to see that every such sum is a solution to (1).

One final note. The function $f(x) = u(x, 0)$ is fundamental in determining solutions to (1) and (2). We treat it as an implicit boundary condition in the applications below.

1 Basics revisited

Here is a more rigorous mathematical treatment of the analysis of the heat equation given in §10.2. The statement

$$\frac{T'(t)}{T(t)} = \beta \frac{X''(x)}{X(x)}$$

for all $0 < x < L$ and $t > 0$, which is (4) on page 604, is too restrictive. We can get around it.

Suppose that $L > 0$. We wish to find a *non-zero* solution to (2). We will try to find such a solution of the form

$$u = XT,$$

where $X : [0, L] \longrightarrow \boldsymbol{R}$ and $T : [0, \infty) \longrightarrow \boldsymbol{R}$ are continuous functions, $X''(x)$ exists for $0 < x < L$, and $T'(t)$ exists for $t > 0$. In particular u is continuous on its domain. Our notation for u simply means that

$$u(x, t) = X(x)T(t)$$

for all $0 \leq x \leq L$ and $t \geq 0$.

Observe that (2) becomes

$$T'(t)X(x) = \beta T(t)X''(x) \tag{3}$$

for all $0 < x < L$ and $t < 0$.

By assumption $u \neq 0$. Continuity of u means $X(x_0)T(t_0) = u(x_0, t_0) \neq 0$ for some $0 < x_0 < L$ and $t_0 > 0$. In particular $X(x_0) \neq 0$ and $T(t_0) \neq 0$. Since $X(x_0) \neq 0$ we may write

$$T'(t) = \left(\frac{\beta X''(x_0)}{X(x_0)}\right) T(t),$$

or

$$T'(t) = \alpha T(t),$$

where

$$\alpha = \frac{\beta X''(x_0)}{X(x_0)},$$

for all $t > 0$. Therefore there is a real number A that $T(t) = Ae^{\alpha t}$ for all $t > 0$. Since $0 \neq T(t_0)$ it follows that $A \neq 0$. Consequently $T(t) \neq 0$ for all $t > 0$ and

$$\frac{T'(t)}{T(t)} = \alpha = \frac{T'(t_0)}{T(t_0)} \tag{4}$$

for all $t > 0$.

Since $T(t_0), \beta \neq 0$ we may define

$$K = \frac{T'(t_0)}{\beta T(t_0)}.$$

By (4) we have the relation

$$T'(t) = \beta K T(t) \tag{5}$$

for all $t > 0$. Since $T'(t_0)X(x) = \beta T(t_0)X''(x)$ by (3), we conclude that

$$X''(x) = KX(x) \tag{6}$$

for all $0 < x < L$.

Solution to (5): $T(t) = Ae^{\beta Kt}$ for some constant A. Since $T(t_0) \neq 0$ it follows that $A \neq 0$.

Solution to (6): We rewrite (6) as $X'' - KX = 0$. The auxiliary equation for this second order linear homogeneous differential equation is $r^2 + K = 0$. There are three cases to consider.

Case 1: $K > 0$. $X(x) = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$ is the general solution; a fundamental solution set is $\{e^{\sqrt{K}x},\ e^{-\sqrt{K}x}\}$.

Case 2: $K = 0$. $X(x) = c_1 e^{0x} + c_2 x e^{0x} = c_1 + c_2 x$ is the general solution; a fundamental solution set is $\{1,\ x\}$.

Case 3: $K < 0$. $X(x) = c_1 \cos\sqrt{|K|}x + c_2 \sin\sqrt{|K|}x$ is the general solution; a fundamental solution set is $\{\cos\sqrt{|K|}x,\ \sin\sqrt{|K|}x\}$.

2 Boundary conditions for the heat equation - ends of rod have constant temperature 0

This translates to

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

with boundary condition

$$u(0,t) = 0 = u(L,t), \tag{7}$$

or equivalently

$$X(0)T(t) = 0 = X(L)T(t),$$

for all $t > 0$. Since $T(t_0) \neq 0$ for some $t_0 > 0$, the boundary condition is equivalent to

$$X(0) = 0 = X(L). \tag{8}$$

We run through the cases above to see which produce a non-zero solution.

Case 1: $K > 0$. Here

$$\begin{aligned} c_1 e^{\sqrt{K}0} + c_2 e^{-\sqrt{K}0} &= X(0) = 0 \\ c_1 e^{\sqrt{K}L} + c_2 e^{-\sqrt{K}L} &= X(L) = 0 \end{aligned}$$

which translates to

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\sqrt{K}L} + c_2 e^{-\sqrt{K}L} &= 0. \end{aligned}$$

From the first equation we deduce that $c_2 = -c_1$ and thus the second becomes

$$c_1(e^{\sqrt{K}L} - e^{-\sqrt{K}L}) = 0.$$

Since the exponential function e^x is one-one, and $\sqrt{K}L \neq -\sqrt{K}L$ as the former is a positive number, it follows that $e^{\sqrt{K}L} \neq e^{-\sqrt{K}L}$. We have shown that $e^{\sqrt{K}L} - e^{-\sqrt{K}L} \neq 0$. Therefore $c_2 = -c_1 = 0$ and no non-zero solution arises in this case.

Case 2: $K = 0$. Here

$$\begin{aligned} c_1 + c_2 0 &= X(0) &= 0 \\ c_1 + c_2 L &= X(L) &= 0 \end{aligned}$$

which means that $c_1 = 0 = c_2 L$. Since $L > 0$ necessarily $c_2 = 0$ as well. This case gives rise to the zero solution only.

Case 3: $K < 0$. Here

$$\begin{aligned} c_1 \cos\sqrt{|K|}0 + c_2 \sin\sqrt{|K|}0 &= X(0) &= 0 \\ c_1 \cos\sqrt{|K|}L + c_2 \sin\sqrt{|K|}L &= X(L) &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} c_1 &= 0 \\ c_2 \sin\sqrt{|K|}L &= 0. \end{aligned}$$

A non-zero solution to arise from this case if and only if

$$\sin\sqrt{|K|}L = 0, \tag{9}$$

or equivalently

$$|K|L = n\pi$$

for some integer n. Since $\sqrt{|K|}L$ is positive, and $K < 0$, the solutions to (9) are described by

$$K = -(\frac{n\pi}{L})^2$$

where n is a positive integer. Linear combinations of these solutions for various n's is a solution to (2). This suggests the formal "solution"

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin \frac{n\pi}{L} x,$$

where $0 \le x \le L$ and $t \ge 0$, which indeed is a solution to (2) under reasonable circumstances. Note that

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

for $0 \le x \le L$.

3 Boundary conditions for the heat equation - ends of rod insulated

This translates to

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

and the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t),$$

or equivalently

$$X'(0)T(t) = 0 = X'(L)T(t),$$

for all $t > 0$. Since $T(t_0) \neq 0$ for some $t_0 > 0$, the boundary condition is equivalent to

$$X'(0) = 0 = X'(L). \tag{10}$$

for all $t > 0$. Compare with (8).

We run through the cases above to see which produce a non-zero solution.

Case 1: $K > 0$. Here

$$\begin{aligned} c_1(\sqrt{K}e^{\sqrt{K}0}) + c_2(-\sqrt{K}e^{-\sqrt{K}0}) &= X'(0) &= 0 \\ c_1(\sqrt{K}e^{\sqrt{K}L}) + c_2(-\sqrt{K}e^{-\sqrt{K}L}) &= X'(L) &= 0 \end{aligned}$$

which translates to

$$\begin{aligned} (c_1 - c_2)\sqrt{K} &= 0 \\ (c_1 e^{\sqrt{K}L} - c_2 e^{-\sqrt{K}L})\sqrt{K} &= 0. \end{aligned}$$

Now $\sqrt{K} \neq 0$. Thus from the first equation we deduce that $c_2 = c_1$ and consequently the second becomes

$$c_1(e^{\sqrt{K}L} - e^{-\sqrt{K}L}) = 0$$

For reasons cited above $c_2 = c_1 = 0$. Therefore no non-zero solution arises from this case.

Case 2: $K = 0$. Here

$$\begin{aligned} c_2 &= X'(0) &= 0 \\ c_2 L &= X'(L) &= 0 \end{aligned}$$

which holds if and only if $c_2 = 0$. Thus $X(x) = c_1$ is a solution in this case, where c_1 is any real number.

Case 3: $K < 0$. Here

$$\begin{aligned} c_1(-\sqrt{|K|}\sin\sqrt{|K|}0) + c_2(\sqrt{|K|}\cos\sqrt{|K|}0) &= X'(0) &= 0 \\ c_1(-\sqrt{|K|}\sin\sqrt{|K|}L) + c_2(\sqrt{|K|}\cos\sqrt{|K|}L) &= X'(L) &= 0 \end{aligned}$$

which, since $K \neq 0$, is equivalent to

$$\begin{aligned} c_2 &= 0 \\ c_1 \sin\sqrt{|K|}L &= 0. \end{aligned}$$

Since $c_2 = 0$, a non-zero solution to arise from this case if and only if (9) holds; that is

$$K = -(\frac{n\pi}{L})^2$$

where n is a positive integer. Linear combinations of these solutions for various n's, together with the solution when $K = 0$, is a solution to (2). This suggests the formal "solution"

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(\frac{n\pi}{L})^2 t} \cos \frac{n\pi}{L} x,$$

where $0 \le x \le L$ and $t \ge 0$, which indeed is a solution to (2) under reasonable circumstances. (Here we write $X(x) = \frac{a_0}{2}$ when $K = 0$.) Note that

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

for $0 \le x \le L$.

4 Boundary conditions for the heat equation - ends of rod have constant temperatures

We wish to solve

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

with the boundary condition

$$u(0,t) = U_1 \qquad \text{and} \qquad u(L,t) = U_2 \tag{11}$$

for all $t > 0$. We have noted that if u_1, u_2 are solutions to (2) then $c_1 u_1 + c_2 u_2$ is as well for any real numbers c_1, c_2.

Now suppose u, u_p satisfy (2) and the boundary condition (11). Then $u_h = u - u_p$ satisfies (2). Since

$$u_h(0,t) = u(0,t) - u_p(0,t) = U_1 - U_1 = 0$$

and

$$u_h(L,t) = u(L,t) - u_p(L,t) = U_2 - U_2 = 0$$

for all $t > 0$, it follows that u_h satisfies the boundary condition (7) of Section 2. Here u_p and u_h do not have the same meaning as in our discussion at the beginning of these notes – but there are obvious parallels. Note that $u = u_p + u_h$. We leave the reader with the exercise of showing that if u_p is a solution to (2) which satisfies (11), then the solutions to (2) which satisfy (11) are the $u = u_p + u_h$, where u_h satisfies (2) and (7).

We have described a class of solutions to (2) which satisfy (7) in Section 2, namely

$$u_h = \sum_{n=1}^{\infty} b_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin \frac{n\pi}{L} x.$$

To complete our discussion of this case we will find a particular u_p.

Some motivation. Suppose that $u_p = XT$ as in Section 2. Then (11) translates to

$$X(0)T(t) = U_1 \qquad \text{and} \qquad X(L)T(t) = U_2$$

for all $t > 0$. Thus if one of U_1, U_2 is not zero, $T = c$ is *constant* on $(0, \infty)$. This suggests replacing $u = Xc$ by X, in which case $T = 1$. Thus we will assume

$$u_p(x,t) = X(x)$$

for all $0 \leq x \leq L$ and $t \geq 0$. In this case (11) translates to $0 = \beta X''(x)$, or equivalently $X''(x) = 0$, for all $0 < x < L$ since $\beta \neq 0$. Let us assume that X is continuous on $[0, L]$. Then there are real numbers a, b such that $X(x) = a + bx$ for all $0 \leq x \leq L$. Note that (11) translates to

$$\begin{aligned} a &= u_p(0,t) = U_1 \\ a + bL &= u_p(L,t) = U_2, \end{aligned}$$

or equivalently

$$u_p(x,t) = U_1 + \frac{U_2 - U_1}{L} x$$

for all $0 \leq x \leq L$ and $t > 0$. Thus

$$u(x,t) = u_p(x,t) + u_h(x,t) = U_1 + \frac{U_2 - U_1}{L} x + \sum_{n=1}^{\infty} b_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin \frac{n\pi}{L} x$$

provides a formal solution to (2) with the boundary condition (11). Observe that

$$u(x,0) = u_p(x,0) + u_h(x,0) = U_1 + \frac{U_2 - U_1}{L}x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x.$$

5 The heat equation - constant source present

We wish to solve

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x), \tag{12}$$

that is (1) where $P(x,t) = P(x)$ does not depend on t, with the boundary condition

$$u(0,t) = U_1 \qquad \text{and} \qquad u(L,t) = U_2 \tag{13}$$

for all $t > 0$. Let u_p be a particular solution to (12) . We have noted that the solutions to (1) are $u = u_p + u_h$, where u_h is a solution to (2). Taking the boundary condition (13) into account, suppose that u_p satisfies (13) also. Then the solutions to (12) which satisfy the boundary condition (13) are those of the form $u = u_p + u_h$ above, where u_h satisfies the boundary condition (7) of Section 2. Again, a formal solution for u_h is

$$u_h = \sum_{n=1}^{\infty} b_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin \frac{n\pi}{L}x.$$

It remains to find a particular solution u_p.

As in the previous section, we suppose that $u_p = X$. Then (12) and the boundary condition (13) translate to

$$X''(x) = -\frac{1}{\beta}P(x) \tag{14}$$

and

$$X(0) = U_1 \quad \text{and} \quad X(L) = U_2 \tag{15}$$

respectively. Suppose that X_p and X are solutions to (14). Now $X_h = X - X_p$ satisfies $X_h'' = 0$. Assuming that $X_h(x)$ is continuous on $[0,L]$, there are real numbers a, b such that

$$X_h(x) = a + bx$$

for all $0 \le x \le L$. Therefore

$$X(x) = X_p(x) + X_h(x) = X_p(x) + a + bx$$

for all $0 \le x \le L$. *Without loss of generality we may assume* $X_p(0) = 0$. From the equations

$$\begin{aligned} X_p(0) + a + b0 &= X(0) &= U_1 \\ X_p(L) + a + bL &= X(L) &= U_2 \end{aligned}$$

we deduce

$$a = U_1 - X_p(0) = U_1$$

and thus

$$b = \frac{U_2 - X_p(L) - U_1}{L}.$$

Therefore

$$u_p(x) = X(x) = X_h(x) + X_p(x) = \left(U_1 + \frac{U_2 - U_1 - X_p(L)}{L} x\right) + X_p(x)$$

for all $0 \le x \le L$. Since $u = u_p + u_h = X + u_h$ we finally have

$$u(x,t) = U_1 + \frac{U_2 - U_1 - X_p(L)}{L} x + X_p(x) + \sum_{n=1}^{\infty} b_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin \frac{n\pi}{L} x$$

and consequently

$$u(x,0) = U_1 + \frac{U_2 - U_1 - X_p(L)}{L} x + X_p(x) + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x.$$

We can take

$$X_p(x) = -\int_0^x \left(\int_0^z \frac{1}{\beta} P(s)\, ds\right) dz;$$

for $X_p(0) = 0$ and by the Fundamental Theorem of Calculus $X_p(x)$ satisfies $X_p''(x) = -\frac{1}{\beta} P(x)$.