MATH 330, MTHT 435 Final Examination **Radford** **12/08/03**

Name (print) ____________________________________

(1) There are *eight questions* on this exam. (2) you may *keep* this exam copy. (3) *Write* your solutions to problems in your exam booklet. (4) You need *not* show how associativity is used in proofs. (5) $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote the integers, rational numbers, real numbers, and complex numbers respectively. (6) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (30 pts.) Let $G = <a>$ be a cyclic group of order 66.

   a) How many subgroups does G have?

   b) For each subgroup of G list its size and *one* generator.

   c) List *all* generators of the subgroup of G of order 6.

   d) Find a divisor d of 66 such that $<a^{-42}> = <a^d>$ and list the distinct elements of $<a^{-42}>$.

   e) Write down the lattice of all subgroups of G.

2. (20 pts.) Consider the permutation $f = (1\ 3\ 2\ 4\ 9\ 8\ 6\ 5\ 7)(4\ 7)(6\ 9)$ of S_9.

   a) Write f as a product of *disjoint* cycles.

   b) Write f as a product of transpositions.

   c) Is f even? You must justify your answer.

   d) What is the order of f?

3. (25 pts.) Let $G = \{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \boldsymbol{R} \}$.

   a) Show that G is a subgroup of $\mathrm{GL}(3, \boldsymbol{R})$.

   b) Show that $\boldsymbol{R} \oplus \boldsymbol{R} \simeq G$, where $\boldsymbol{R}$ is regarded as a group under addition.

4. (25 pts.) Let G, G' be finite groups where $G = <a>$ is cyclic of order n.

a) For $b \in G'$, show that $f : G \longrightarrow G'$ given by $f(a^m) = b^m$ is a well-defined group homomorphism if and only if $|b|$ divides n.

b) Find all group homomorphisms $f : \boldsymbol{Z}_{20} \longrightarrow \boldsymbol{Z}_{32}$.

5. (25 pts.) Consider ring $R = \boldsymbol{Z}_{12}$.

a) List the *nilpotent* elements of R.

b) List the *zero divisors* of R. For each zero divisor a list a non-zero $b \in R$ such that $ab = 0$.

c) List the units of R and write down a Cayley table for $U(R)$.

6. (25 pts.) Consider the subset $R = \{a + b\imath \mid a, b \in \boldsymbol{Z}\}$ of the field of complex numbers $\boldsymbol{C}$.

a) Show that R is a subring of $\boldsymbol{C}$.

b) Determine whether or not R a subfield of $\boldsymbol{C}$.

c) List the elements of $U(R)$. Is $U(R)$ a cyclic group? Justify your answer. [Hint: If z is a non-zero complex number then $1 = |zz^{-1}| = |z||z^{-1}|$, and thus $1 = |z|^2|z^{-1}|^2$.]

7. (25 pts.) Let $\phi : \boldsymbol{C} \longrightarrow \mathrm{M}(2, \boldsymbol{R})$ be the one-one function from the field of complex numbers to the ring of 2×2 matrices with real coefficients defined by $\phi(a + b\imath) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

a) Show that ϕ is a ring homomorphism.

b) Compute $(3 + 2\imath)^{-1}$. Use the answer and ϕ to find $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}^{-1}$.

c) Compute $(3 + 2\imath)^2$. Use the answer and ϕ to find $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}^{2}$.

8. (25 pts.) Let R be a ring.

a) Define *ideal of* R.

b) Suppose that I, J are ideals of R. Show that $I \cap J$ is an ideal of R.

c) Let $R = \boldsymbol{R}[x]$, $I = <(x+1)(x-3)>$, and $J = <(x-3)(x+5)>$. Find an $f(x) \in \boldsymbol{R}[x]$ such that $I \cap J = <f(x)>$. Justify your choice. [Hint: Recall that if $a_1, \ldots, a_r \in \boldsymbol{R}$ are distinct roots of $f(x) \in \boldsymbol{R}[x]$ then $(x - a_1) \cdots (x - a_r)$ divides $f(x)$.]