MATH 330, MTHT 435 Hour Exam I Solution **Radford**

10/14/03

Name (print) ________________________________

(1) There are *five questions* on this exam. (2) You may *keep* this exam copy. (3) You do *not* need to show how associativity is used in proofs. (4) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (20 pts.) Let G be a group and suppose that H, K are subgroups of G. Show that $H \cap K$ is a subgroup of G.

   There are various approaches. Our first solution is very direct.

   Solution 1: $H \cap K \neq \emptyset$ (**5 points**) *and* $e \in H \cap K$ (**5 points**). Since H, K are subgroups of G it follows that $e \in H, K$. Therefore $e \in H \cap K$ and thus $H \cap K \neq \emptyset$. *Closure* (**5 points**). Suppose that $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H, K are subgroups of G it follows that $ab \in H, K$. Therefore $ab \in H \cap K$. *Inverses* (**5 points**). Let $a \in H \cap K$. Then $a \in H, K$, and since both are subgroups of G we conclude that $a^{-1} \in H, K$. Therefore $a^{-1} \in H \cap K$.

   Solution 2: $H \cap K \neq \emptyset$ (**5 points**). As above, $e \in H \cap K$. *One Step Subgroup Test* (**15 points**). We have established that $H \cap K \neq \emptyset$. Suppose that $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H, K are subgroups of G it follows that $ab^{-1} \in H, K$ by the One Step Subgroup Test. Therefore $ab^{-1} \in H \cap K$. Again by the One Step Subgroup Test $H \cap K$ is a subgroup of G.

2. (20 pts.) Let G be a group.

   a) Suppose that $a, b \in G$. Show that $(ab)^{-1} = b^{-1}a^{-1}$.

   Solution 1: We will verify that ab and $b^{-1}a^{-1}$ are inverses using associativity.

   $$(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$$

   and

   $$(b^{-1}a^{-1})(ab) = b^{-1}a^{-1}ab = b^{-1}eb = b^{-1}b = e.$$

   Therefore by definition $b^{-1}a^{-1}$ is an inverse of ab. (**8 points**)

   Solution 2: Many of you used cancelation. Since

   $$(ab)(ab)^{-1} = e \qquad \text{and} \qquad (ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aea^{-1} = a^{-1}a = e$$

   we have

   $$(ab)(ab)^{-1} = (ab)(b^{-1}a^{-1})$$

   and therefore $(ab)^{-1} = b^{-1}a^{-1}$ by (left) cancelation.

b) Suppose that n is a positive integer and $a_1, \ldots, a_n \in G$. Show, by induction, that $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$.

A Solution: Induction is to be used. The case $n = 1$ holds since $a_1^{-1} = a_1^{-1}$. Suppose that $n \geq 1$ and the inverse formula holds for $a_1, \ldots, a_n \in G$. Let $a_1, \ldots, a_{n+1} \in G$. Then using part a) and our induction hypothesis we calculate

$$(a_1 \cdots a_{n+1})^{-1} = ((a_1 \cdots a_n)a_{n+1})^{-1} = a_{n+1}^{-1}(a_1 \cdots a_n)^{-1} = a_{n+1}^{-1}(a_n^{-1} \cdots a_1^{-1}).$$

Thus

$$(a_1 \cdots a_{n+1})^{-1} = a_{n+1}^{-1}a_n^{-1} \cdots a_1^{-1}$$

and our product formula holds for $a_1, \ldots, a_{n+1}$. (**12 points**)

3. (25 pts.) Let $G = <a>$ be a cyclic group of order 30.

a) How many subgroups does G have?

Solution: The correspondence $H \mapsto |H|$ describes a bijection between the set of subgroups of G and the set of divisors of $|G| = 30$. Since the divisors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$ we conclude that G has 8 subgroups. (**6 points**)

b) For each subgroup of G list its size and *one* generator.

Solution:

size	a generator
1	$a^{30} = e$
2	a^{15}
3	a^{10}
5	a^6
6	a^5
10	a^3
15	a^2
30	$a^1 = a$

(**7 points**).

c) List *all* of the generators of the subgroup of G of order 15.

Solution: The integers $1 \leq k \leq 15$ which are relatively prime to 15 are: $1, 2, 4, 7, 8, 11, 13, 14$. Thus the generators of the subgroup of order 15 of G are

$$(a^2)^1 = a^2, \quad (a^2)^2 = a^4, \quad (a^2)^4 = a^8, \quad (a^2)^7 = a^{14},$$
$$(a^2)^8 = a^{16}, \quad (a^2)^{11} = a^{22}, \quad (a^2)^{13} = a^{26}, \quad (a^2)^{14} = a^{28}.$$ (**6 points**)

d) Find a divisor d of 30 such that $<a^{25}> = <a^d>$ and list the distinct elements of $<a^{25}>$.

Solution: The greatest common divisor of 25 and 30 is 5. Therefore we may take $d = 5$ (**3 points**) and thus $<a^{25}> = <a^5> = \{e, a^5, a^{10}, a^{15}, a^{20}, a^{25}\}$. (**3 points**). *Note*: $d = -5$ works just as well.

4. (20 pts.) Consider the permutation $f = (7\,6\,9)(1\,3\,4\,5\,2)(2\,3)(7\,8)$ of S_9.

a) Write f as a product of *disjoint* cycles.

Solution: $f = (1\ 3)(2\ 4\ 5)(6\ 9\ 7\ 8)$, or any rearrangement of these cycles. (**7 points**)

b) Write f as a product of transpositions (2-cycles).

Solutions: Using our cyclic decomposition we have $f = (1\ 3)(2\ 5)(2\ 4)(6\ 8)(6\ 7)(6\ 9)$ (**8 points**). One can go back and replace the cycles in the definition of f and write

$$f = (7\ 9)(7\ 6)(1\ 2)(1\ 5)(1\ 4)(1\ 3)(2\ 3)(7\ 8)$$

as well.

c) Is f even? You must justify your answer.

Solution: Since f is the product of an even number of 2-cycles it follows that f is *even*. (**5 points**)

5. (15 pts.) Let $G = S_8$.

a) Construct a permutation $f \in G$ of order 15.

Solution: $f = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$ for example. (**7 points**)

b) Show that there is no permutation in G of order 14.

Solution: Suppose that $f \in S_8$ has order 14 and write f as a product of disjoint cycles. Since the order of f is the least common multiple of its cycle lengths, it follows that f has a cycle whose length is divisible by 7. Since $f \in S_8$ necessarily f has two cycles, one of length 7 and one of length 1. Thus means that the order of f is 7, a contradiction. Therefore no permutation in S_8 has order 14. (**8 points**)