MATH 330, MTHT 435 Final Examination **Radford** **12/06/04**

Name (print) ________________________________

(1) There are *eight questions* on this exam. (2) you may *keep* this exam copy. (3) *Write* your solutions to problems in your exam booklet. (4) You need *not* show how associativity is used in proofs. (5) $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote the integers, rational numbers, real numbers, and complex numbers respectively. (6) *You are expected to abide by the University's rules concerning academic honesty.*

---

1. (30 pts.) Let $G = <a>$ be a cyclic group of order 75.

   a) How many subgroups does G have?

   b) For each subgroup of G list its size and *one* generator in the form a^ℓ, where $0 \leq \ell < 75$.

   c) List *all* generators of the subgroup of G of order 15 in the form a^ℓ, where $0 \leq \ell < 75$.

   d) Find a divisor d of 75 such that $<a^{-400}> = <a^d>$ and list the distinct elements of $<a^{-400}>$ as a^ℓ, where $0 \leq \ell < 75$.

   e) Write down the lattice of all subgroups of G.

2. (20 pts.) Consider the permutation $f = (1\ 6\ 3\ 4)(2\ 5\ 7)(1\ 6\ 5)(5\ 6\ 10\ 9\ 8)$ of S_{10}.

   a) Write f as a product of *disjoint* cycles.

   b) Write f as a product of transpositions.

   c) Is f even? You must justify your answer.

   d) What is the order of f?

3. (25 pts.) Let $G = <a>$ be a finite cyclic group and suppose that $f : G \longrightarrow G'$ is a group homomorphism.

   a) Show that $f(a)$ has finite order and $|f(a)|$ divides $|a|$.

   b) Suppose that $b \in G'$ has finite order and $|b|$ divides $|a|$. Show that the rule $\phi : G \longrightarrow G'$ given by $\phi(a^\ell) = b^\ell$ for all $\ell \in \mathbf{Z}$ is a *well-defined* group homomorphism. (Well-defined means that $a^\ell = a^{\ell'}$ implies $b^\ell = b^{\ell'}$.)

[Hint: For parts a) and b) you may use the following: If G is any group and $g \in G$ has finite order, then $g^m = e$ if and only if $|g|$ divides m.]

c) Suppose that $|G| = 60$ and $G' = <c>$ is cyclic of order 175. Determine all group homomorphisms $\phi : G \longrightarrow G'$.

4. (25 pts.) Consider the subgroup $H = <(1\ 2\ 4)>$ of A_4.

 a) What is the number of left cosets of H in A_4? In S_4?

 b) Write the elements of $(1\ 3)H$ as products of disjoint cycles.

 c) Determine whether or not H is a normal subgroup of S_4.

5. (25 pts.) Consider the group $G = \boldsymbol{Z}_{16} \oplus \mathbf{Z}_6$.

 a) Find all elements of G of order 6.

 b) Find a cyclic subgroup of G of order 12 and list its elements.

 c) Find a non-cyclic subgroup of G of order 12.

 d) Now regard $R = \boldsymbol{Z}_{16} \oplus \mathbf{Z}_6$ as a ring. Given that $U(R) = U(\mathbf{Z}_{16}) \oplus U(\mathbf{Z}_6)$, find the *order* of $U(R)$.

6. (25 pts.) Consider the subset $R = \{m + n\sqrt{5} \mid m, n \in \boldsymbol{Z}\}$ of the field of real numbers $\mathbf{R}$.

 a) Determine whether or not R is an additive subgroup of $\mathbf{R}$.

 b) Determine whether or not R is a subring of $\mathbf{R}$.

 c) Determine whether or not R is a subfield of $\mathbf{R}$.

[Hint: You may use the fact that $r + s\sqrt{5} = r' + s'\sqrt{5}$, where $r, r', s, s' \in \mathbf{Q}$, implies that $r = r'$ and $s = s'$. This follows from the fact that $\sqrt{5}$ is not rational.]

7. (25 pts.) Let R be a ring. for non-empty subsets $S_1, \ldots, S_n$ of R define

$$S_1 + \cdots + S_n = \{s_1 + \cdots + s_n \mid s_i \in S_i \text{ for all } 1 \leq i \leq n\}.$$

 a) Define ideal of R.

 b) Suppose that I, J are ideals of R. Show that $I + J$ is an ideal of R.

 c) Suppose that $I_1, \ldots, I_n$ are ideals of R. Show, by induction, that $I_1 + \cdots + I_n$ is an ideal of R.

8. (25 pts.) Let R be a ring with unity 1 and suppose that I is an ideal of R.

 a) Show that $1 \in I$ implies $I = R$.

 b) Now let $R = \mathbf{R}[x]$ and suppose that $x^3 - 6x - 1, x^2 + 3x + 1 \in I$. Show that $I = R$. [Hint: You might find the Division Algorithm useful in finding other elements in I].