Homework #14 (week of 11/22–11/24)

Due Wednesday, 11/24/04 in class. REVISED 11/12/04

Suppose that G is an (additive) abelian group and $S_1, \ldots, S_n$ are non-empty subsets of G. Then

$$S_1 + \cdots + S_n = \{s_1 + \cdots + s_n \mid s_\imath \in S_\imath \text{ for all } 1 \leq \imath \leq n\}.$$

You may use the associative, distributive, and commutative laws (the latter for addition only) for rings without particular reference.

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1. Let R be a ring.

a) Suppose that $\{I_s\}_{s \in S}$ is an indexed family of ideals of R. Show that $I = \cap_{s \in S} I_s$ is an ideal of R.

b) Let $I_1, \ldots, I_n$ be ideals of R. Show that $I_1 + \cdots + I_n$ is an ideal of R.

2. Let R be a ring. A *left ideal of* R is subspace L of R which satisfies $ra \in L$ for all $r \in R$ and $a \in L$.

a) Suppose that L is a left ideal of R and $a \in R$. Show that

$$La = \{ra \mid r \in L\}$$

is a left ideal of R. (In particular Ra is a left ideal of R.)

b) Suppose that $L_1, \ldots, L_n$ are left ideals of R. Show that $L_1 + \cdots + L_n$ is a left ideal of R. (In particular $Ra_1 + \cdots + Ra_n$ is an ideal of R for all $a_1, \ldots, a_n \in R$.)

3. Let $R_1, \ldots, R_n$ be rings with unity and $R = R_1 \oplus \cdots \oplus R_n$.

a) Suppose that $I_1, \ldots, I_n$ are ideals of $R_1, \ldots, R_n$ respectively. Show that $I_1 \oplus \cdots \oplus I_n$ is an ideal of R.

b) Suppose that $1 \leq \imath \leq n$ is fixed. Show that $f : R \longrightarrow R_\imath$ defined by $f((a_1, \ldots, a_n)) = a_\imath$ for all $(a_1, \ldots, a_n) \in R$ is an onto ring homomorphism..

c) Suppose that I is an ideal of R. Show that $f_\imath(I)$ is an ideal of $R_\imath$ for all $1 \leq \imath \leq n$ and that $I = f_1(I) \oplus \cdots \oplus f_n(I)$. [Hint: Note that $x = (f_1(x), \cdots, f_n(x))$ for all $x \in R$. Show that $I \subseteq f_1(I) \oplus \cdots \oplus f_n(I) = J$.

Conversely, if $y \in J$ then $y = (f_1(x_1), \ldots, f_n(x_n))$ for some $x_1, \ldots, x_n \in I$. Let $e_\imath = (0, \ldots, 1, \ldots, 0) \in R$ be the n-tuple with entries zero with one exception which has value 1 and is in the $\imath^{th}$ position and let $x = x_1e_1 + \cdots x_ne_n$. Show $y = (f_1(x), \cdots, f_n(x)) = x$.]

4. Let $R = \mathbf{Z} \oplus \mathbf{Z}$.

a) Determine all prime ideals of R.

b) Determine all maximal ideals of R.