Solution to Homework #14 (week of 11/22–11/24)

12/02/04 Radford

Suppose that G is an (additive) abelian group and $S_1, \ldots, S_n$ are non-empty subsets of G. Then

$$S_1 + \cdots + S_n = \{s_1 + \cdots + s_n \,|\, s_\imath \in S_\imath \text{ for all } 1 \leq \imath \leq n\}.$$

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1. (**10 points total**) R is a ring.

a) The indexed family of ideals $\{I_s\}_{s \in S}$ is an indexed family of (additive) subgroups of R. We have shown that $I = \cap_{s \in S} I_s$ is an (additive) subgroup of R. (**3 points**)

Let $r \in R$ and $a \in I$. Then $a \in I_s$ for all $s \in S$. Let $s \in S$. Then I_s is an ideal of R. Therefore $ra, ar \in I_s$. Consequently $ra, ar \in \cap_{s \in S} I_s = I$. We have shown that I is an ideal of R. (**2 points**)

b) Let $I = I_1 + \cdots + I_n$. By assumption $I_1, \ldots, I_n$ are ideals of R. Since they are also additive subgroups of R it follows that $I_\imath \neq \emptyset$ for all $1 \leq \imath \leq n$. Therefore $I = I_1 + \cdots + I_n \neq \emptyset$. (**1 points**)

Let $x, y \in I$. Then $x = a_1 + \cdots + a_n$ and $y = b_1 + \cdots + b_n$ where $a_\imath, b_\imath \in I_\imath$ for all $1 \leq \imath \leq n$. Let $1 \leq \imath \leq n$. Since $I_\imath$ is a additive subgroup $a_\imath - b_\imath \in I_\imath$. Therefore

$$x - y = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) = (a_1 - b_1) + \cdots (a_n - b_n) \in I_1 + \cdots + I_n$$

which shows that I is an additive subgroup of R by the 1-Step Subgroup Test. (**2 points**)

Now let $r \in R$ and let $1 \leq \imath \leq n$. Since $I_\imath$ is an ideal of R we have $ra_\imath, a_\imath r \in I_\imath$. Therefore

$$rx = r(a_1 + \cdots + a_n) = ra_1 + \cdots + ra_n \in I_1 + \cdots + I_n = I$$

and

$$xr = (a_1 + \cdots + a_n)r = a_1r + \cdots + a_nr \in I_1 + \cdots + I_n = I. \ (\mathbf{2\ points})$$

This completes our proof that I is an ideal of R.

2. (**10 points total**) R is a ring, $a \in R$, and $La = \{ra \,|\, r \in L\}$.

a) By assumption L is a left ideal of R and $a \in R$. Since L is a additive subgroup of R we have $L \neq \emptyset$. Thus $La \neq \emptyset$.

Suppose that $x, y \in La$. Then $x = sa$ and $y = ta$ for some $s, t \in L$. Since L is an additive subgroup of R it follows that $s - t \in L$ by the 1-Step Subgroup Test. Therefore

$$x - y = sa - ta = (s - t)a \in La.$$

by the 1-Step Subgroup Test La is an additive subgroup of R.

Let $r \in R$. Then

$$rx = r(sa) = (rs)a \in La$$

since L is a left ideal of R. Therefore La is a left ideal of R. (**5 points**)

b) Suppose that $L_1, \ldots, L_n$ are left ideals of R. Our argument for part b) of the preceding exercise showed that if $A_1, \ldots, A_n$ are additive subgroups of R then $A_1 + \cdots + A_n$ is an additive subgroup of R. Thus $L = L_1 + \cdots + L_n$ is an additive subgroup of R.

Suppose that $r \in R$ and $x \in L$. Then $x = a_1 + \cdots + a_n$ where $a_\imath \in L_\imath$ for all $1 \leq \imath \leq n$. Fix $1 \leq \imath \leq n$. Since $L_\imath$ is a left ideal of R and $a_\imath \in L_\imath$ it follows that $ra_\imath \in L_\imath$. Thus

$$r(a_1 + \cdots + a_n) = ra_1 + \cdots + ra_n \in L_1 + \cdots + L_n = L.$$

We have shown that L is a left ideal of R. (**5 points**)

3. (**10 points total**) Let $R_1, \ldots, R_n$ be rings with unity and $R = R_1 \oplus \cdots \oplus R_n$.

a) Since $I_1, \ldots, I_n$ are ideals of $R_1, \ldots, R_n$ respectively, they are additive subgroups and hence not empty. Thus $I_1 \oplus \cdots \oplus I_n \neq \emptyset$. Let $x, y \in I_1 \oplus \cdots \oplus I_n$. Then $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$, where $a_\imath, b_\imath \in I_\imath$ for all $1 \leq \imath \leq n$. Thus $a_\imath - b_\imath \in I_\imath$ for all $1 \leq \imath \leq n$ by the 1-Step Subgroup Test. Thus

$$x - y = (a_1, \ldots, a_n) - (b_1, \ldots, b_n) = (a_1 - b_1, \ldots, a_n - b_n) \in I_1 \oplus \cdots \oplus I_n$$

which means that $I_1 \oplus \cdots \oplus I_n$ is an additive subgroup of $R_1 \oplus \cdots \oplus R_n$ by the 1-Step subgroup Test. Now let $r \in R_1 \oplus \cdots \oplus R_n$. Then $r = (r_1, \ldots, r_n)$ where $r_\imath \in R_\imath$for all $1 \leq \imath \leq n$. Let $1 \leq \imath \leq n$. Since $I_\imath$ is an ideal of $R_\imath$ the products $r_\imath a_\imath, a_\imath r_\imath \in R_\imath$. Therefore

$$rx = (r_1, \ldots, r_n)(a_1, \ldots, a_n) = (r_1 a_1, \ldots, r_n a_n) \in I_1 \oplus \ldots \oplus I_n$$

and

$$xr = (a_1, \ldots, a_n)(r_1, \ldots, r_n) = (a_1 r_1, \ldots, a_n r_n) \in I_1 \oplus \ldots \oplus I_n.$$

This completes our proof that $I_1 \oplus \ldots \oplus I_n$ is an ideal of $R_1 \oplus \ldots \oplus R_n$. (**3 points**)

b) Let $1 \leq \imath \leq n$ Then $f_\imath : R \longrightarrow R_\imath$ defined by $f((a_1, \ldots, a_n)) = a_\imath$ for all $(a_1, \ldots, a_n) \in R$ is onto. For let $a \in I_\imath$. Then $f_\imath((0, \ldots, a, \ldots, 0) = a$, where the "$a$" in the tuple in the $\imath^{th}$ position.

$$\begin{aligned} f_\imath((a_1, \ldots, a_n)(b_1, \ldots, b_n)) &= f_\imath((a_1 b_1, \ldots, a_n b_n)) \\ &= a_\imath b_\imath \\ &= f_\imath((a_1, \ldots, a_n)) f_\imath((b_1, \ldots, b_n)) \end{aligned}$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in R$ shows that $f_\imath$ is a ring homomorphism. (**3 points**)

c) Suppose that I is an ideal of R. Then $f_\imath(I)$ is an ideal of $R_\imath$ for all $1 \leq \imath \leq n$ since $f_\imath$ is an onto ring homomorphism by part b). Let $x = (a_1, \ldots, a_n) \in R$. Then $f_\imath(x) = a_\imath$ which means that $x = (f_1(x), \ldots, f_n(x))$. Therefore $I \subseteq f_1(I) \oplus \cdots \oplus f_n(I)$. Let $J = f_1(I) \oplus \cdots \oplus f_n(I)$. We have shown that $I \subseteq J$. Part c) will follow if $I = J$. To do this we need only show that $J \subseteq I$.

Let $e_\imath = (0, \ldots, 1, \ldots, 0) \in R$ be the n-tuple with entries zero with one exception which has value 1 and is in the $\imath^{th}$ position. Suppose that $y \in f_1(I) \oplus \cdots \oplus f_n(I)$. Then $y = (f_1(x_1), \ldots, f_n(x_n))$, where $x_\imath \in I_\imath$ for all $1 \leq \imath \leq n$. Thus $x = x_1 e_1 + \cdots + x_n e_n \in I$ since I is an ideal of R. Since

$$\begin{aligned} f_\imath(x) &= f_\imath(x_1) f_\imath(e_1) + \cdots + f_\imath(x_\imath) f_\imath(e_\imath) + \cdots + f_\imath(x_n) f_\imath(e_n) \\ &= f_\imath(x_1) 0 + \cdots + f_\imath(x_\imath) 1 + \cdots + f_\imath(x_n) 0 \\ &= f_\imath(x_\imath) \end{aligned}$$

it follows that

$$x = (f_1(x), \ldots, f_n(x)) = (f_1(x_1), \ldots, f_n(x_n)) = y.$$

Therefore $y = x \in I$. We have shown that $J \subseteq I$. (**4 points**)

4. (**10 points total**) $R = \mathbf{Z} \oplus \mathbf{Z}$.

a) Let P be a prime ideal of R. Since R is always a prime ideal of R, we may assume that $P \neq R$. Now $P = P_1 \oplus P_2$ for some ideals P_1, P_2 of $\mathbf{Z}$ by the preceding exercise. We will show that P_1 and P_2 must be prime.

Suppose that $a, b \in \mathbf{Z}$ and $ab \in P_1$. Then

$$(a, 0)(b, 0) = (ab, 0) \in P_1 \oplus P_2 = P$$

means that $(a, 0) \in P$ or $(b, 0) \in P$ since P is prime. Therefore $a \in P_1$ or $b \in P_2$. We have shown that P_1 is prime.

Likewise P_2 is prime. For suppose that $ab \in P_2$. Then

$$(0, a)(0, b) = (0, ab) \in P_1 \oplus P_2 = P$$

means $(0, a) \in P$ or $(0, b) \in P$ since P is prime. Therefore $a \in P_2$ or $b \in P_2$.

We show that, in addition, that $P_1 - \mathbf{Z}$ or $P_2 = \mathbf{Z}$. For

$$(1, 0)(0, 1) = (0, 0) \in P$$

implies that $(1, 0) \in P$ or $(0, 1) \in P$ since P is prime. In the first case

$$(a, 0) = (a, 0)(1, 0) \in P$$

for all $a \in \mathbf{Z}$ since P is an ideal of R. Thus $P_1 = \mathbf{Z}$. In the second case

$$(0, a) = (0, a)(0, 1) \in P$$

again since P is an ideal of R. Thus $P_2 = \mathbf{Z}$.

Suppose that Q is a prime ideal of $\mathbf{Z}$. We leave it as a small exercise that the ideals $Q \oplus \mathbf{Z}$ and $\mathbf{Z} \oplus Q$ are prime ideals of R. Thus the prime ideals of R are

$$Q \oplus \mathbf{Z} \qquad \text{and} \qquad \mathbf{Z} \oplus Q,$$

where Q is a prime ideal of $\mathbf{Z}$. (**5 points**)

b) Maximal ideals are prime. Thus maximal ideals of R have the form $M \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus M$, where M is a prime ideal of $\mathbf{Z}$. Note that any ideal of R which contains $M \oplus \mathbf{Z}$ (respectively $\mathbf{Z} \oplus M$) must have the form $M' \oplus \mathbf{Z}$ (respectively $\mathbf{Z} \oplus M'$), where M' is an ideal of $\mathbf{Z}$. Therefore M must be maximal. (**5 points**)