Solution to Homework # 4 (week of 09/13–09/17)

09/17/04 Radford

1. (**12 points total**)

a) First of all note that $e \in H, K$ since these are subgroups of G. Therefore $e \in H \cap K$ which shows that $H \cap K \neq \emptyset$ (**2 points**). Suppose that $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. By the One-Step Subgroup Test $a^{-1}b \in H,\ K$. Therefore $a^{-1}b \in H \cap K$. Thus $H \cap K$ is a subgroup of G by the One-Step Subgroup Test (**2 points**).

b) Suppose that $n = 1$. Then the intersection is H_1 which is a subgroup of G by assumption (**2 points**). Now let $n \geq 1$ and suppose that $H_1 \cap \cdots \cap H_n$ is a subgroup of G whenever $H_1, \ldots, H_n$ are subgroups of G. Let $H_1, \ldots H_{n+1}$ be subgroups of G. Then $H_1 \cap \cdots \cap H_{n+1} = (H_1 \cap \cdots \cap H_n) \cap H_{n+1}$ is the intersection of two subgroups of G which is a subgroup of G by part a) (**4 points**). Thus part b) follows by induction on n (**2 points**).

2. (**12 points total**) For $G = \boldsymbol{Z}_{12}$:

a)

Subgroup	element list (or equivalent)	
<0>	$\{0\}$	
<1>	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	
<2>	$\{0, 2, 4, 6, 8, 10\}$	
<3>	$\{0, 3, 6, 9\}$	
<4>	$\{0, 4, 8\}$	
<5>	<1>	(**2 points**)
<6>	$\{0, 6\}$	
<7>	<1>	
<8>	<4>	
<9>	<3>	
<10>	<2>	
<11>	<1>	

b)

elements	order
0	1
$1, 5, 7, 11$	12
$2, 10$	6
$3, 9$	4
$4, 8$	3
6	2

(**2 points**)

c) $G = \boldsymbol{Z}_{12}$ is cyclic and is generated by 1 (as well as 5, or 7, or 11). (**2 points**)

For $G = U(12) = \{1, 5, 7, 11\}$:

a)

Subgroup	element list (or equivalent)
<1>	$\{1\}$
<5>	$\{1, 5\}$
<7>	$\{1, 7\}$
<11>	$\{1, 11\}$

(**2 points**)

b)

elements	order
1	1
$5, 7, 11$	2

(**2 points**)

c) $G = U(12)$ is not cyclic by the first table. (**2 points**)

3. (**12 points total**)

a) We assume that $a^2 = e$ for all $a \in G$. Let $a, b \in G$. Then $(ab)^2 = e$ which means that $(ab)(ab) = e$. Thus $a(b(ab)) = e$ from which

$$b(ab) = e(b(ab)) = (aa)(b(ab)) = a(a(b(ab))) = ae = a$$

follows. Therefore $(ba)b = b(ab) = a$ from which

$$ba = (ba)e = (ba)(bb) = ((ba)b)b = ab.$$

We have shown that $ba = ab$ and thus G is abelian. (**4 points**)

b) Let $a \in G$. Suppose that $a = a^{-1}$. Then $a^2 = aa = aa^{-1} = e$. Conversely, suppose that $a^2 = e$. Then $aa = e$ which means that a is an inverse of a by definition. Since inverses are unique, $a = a^{-1}$. (**2 points**)

c) We define a relation on G by $x \sim y$ if and only if $x = y$ or $x = y^{-1}$. We first show that $\sim$ is an equivalence relation on G.

Let $x \in G$. Since $x = x$ by definition $x \sim x$ (**1 point**). Now suppose $x, y \in G$ and $x \sim y$. Then $x = y$, in which case $y = x$, or $x = y^{-1}$, in which case $y = (y^{-1})^{-1} = x^{-1}$. Thus $y \sim x$. (**1 point**)

Let $x, y, z \in G$ and suppose $x \sim y$, $y \sim z$. Suppose $x = y$. Since $y = z$ or $y = z^{-1}$ we conclude that $x = z$ or $x = z^{-1}$. Thus $x \sim z$. Now suppose that $x \neq y$. Since $x \sim y$ necessarily $x = y^{-1}$. Now $y = z$ or $y = z^{-1}$ implies that $y^{-1} = z^{-1}$ or $y^{-1} = (z^{-1})^{-1} = z$. Therefore $x = z^{-1}$ or $x = z$. In either case $x \sim z$ (**2 points**).

Let $x \in G$. Then $[x] = \{y \in G \mid y \sim x\} = \{x, x^{-1}\}$. Thus $|[x]| = 1, 2$. If $x = e$ then $e = e^{-1}$ means that $|[e]| = 1$. Note that $|[x]| = 1$ if and only if $x = x^{-1}$ which is equivalent to $x^2 = e$ by part b). By assumption this equation has a unique solution in G (which must be $x = e$). Since $|G|$ is the sum of the number of elements in the equivalence classes in G it follows that $|G|$ is odd. (**2 points**)

4. (**4 points total**)

a) $<7> = \{7n \mid n \in \boldsymbol{Z}\}$. (**2 points**)

b) $<7> = \{7^n \mid n \in \boldsymbol{Z}\}$. (**2 points**)