

Rough Draft 10/05/04

1. (**14 points total**) We first note that $308 = 4 \cdot 7 \cdot 11$ and $140 = 4 \cdot 5 \cdot 7$.

a) The greatest common divisor of 308 and 140 is 28. As $\langle a^{140} \rangle = \langle a^{\gcd(308, 140)} \rangle$ it follows that $d = 28$. (**2 points**). The order of $H = \langle a^{28} \rangle$ is therefore $308/28 = 11$ (**1 point**).

b) $|H| = |\langle a^{28} \rangle| = 11$. The integers $1 \leq \ell \leq 11$ which are relatively prime to 11 are the ten integers $1, 2, 3, \dots, 10$. Therefore the generators of $H = \langle a^{28} \rangle$ are

$$(a^{28})^1 = a^{28}, (a^{28})^2 = a^{56}, (a^{28})^3 = a^{84}, (a^{28})^4 = a^{112}, (a^{28})^5 = a^{140},$$

$$(a^{28})^6 = a^{168}, (a^{28})^7 = a^{196}, (a^{28})^8 = a^{224}, (a^{28})^9 = a^{252}, (a^{28})^{10} = a^{280}.$$

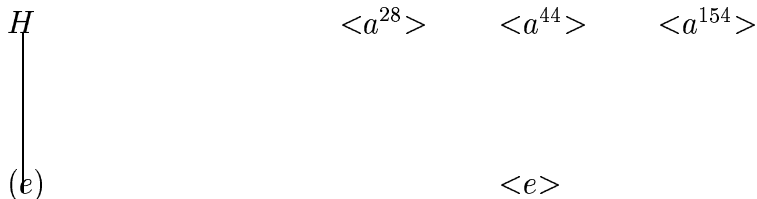
(**3 points**).

c) The divisors of $|H| = 11$ are 1, 11. Thus H has two subgroups; namely $H = \langle (a^{28})^1 \rangle = \langle a^{28} \rangle$ and $\langle (a^{28})^{11} \rangle = \langle a^{308} \rangle = \langle a^0 \rangle$. (**3 points**).

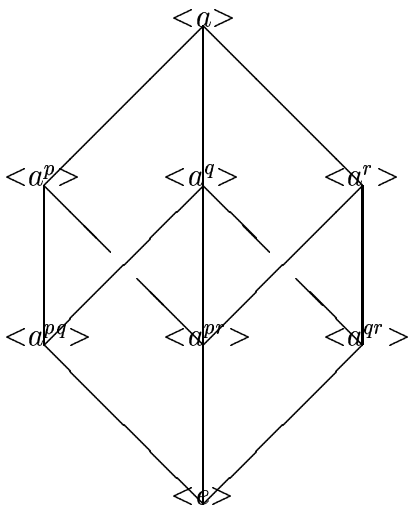
d) (**1 point**) and (**4 points**) respectively. The divisors of 11 are 1, 11. The divisors of 308 are 1, 2, 5, 7, 4, 14, 22, 77, 28, 44, 154, 308. Lines filled in later.
 $\langle a \rangle$

$$\langle a^2 \rangle \quad \langle a^7 \rangle \quad \langle a^{11} \rangle$$

$$\langle a^4 \rangle \quad \langle a^{14} \rangle \quad \langle a^{22} \rangle \quad \langle a^{77} \rangle$$



2. (8 points total)



3. (6 points total) For $\sigma = (1\ 3)(5\ 4\ 6)(4\ 2\ 6)(5\ 2\ 3)(7\ 9\ 2)$ we have:
a) $\sigma = (1\ 3\ 4\ 2\ 7\ 9)(5)(6)$ (1 point); b) $\sigma = (7\ 9)(2\ 9)(4\ 9)(3\ 9)(1\ 9)$ (1 point); and c) σ has order $\text{lcm}(6, 1) = 6$ (1 point).

For $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 1 & 9 & 8 & 2 & 5 & 7 & 3 \end{pmatrix}$ we have: a) $\sigma = (1\ 4\ 9\ 3)(2\ 6)(5\ 8\ 7)$ (1 point); b) $\sigma = ((9\ 3)(4\ 3)(1\ 3))(2\ 6)((8\ 7)(5\ 7))$ (1 point); and c) σ has order $\text{lcm}(4, 2, 3) = 12$ (1 point).

4. (12 points total)

a) By definition of F and τK the set τK consists of the images of K under F . Therefore F is onto (1 point). Suppose that $\sigma, \sigma' \in K$ and $F(\sigma) = F(\sigma')$. Then $\tau\sigma = \tau\sigma'$. By (left) cancelation $\sigma = \sigma'$. Therefore F is one-one. (2 points)

b) We first show that $K \cap \tau K = \emptyset$. Since τ is odd and K consists of even permutations, the set of products τK consists of odd permutations. Since no permutation is both even and odd, $K \cap \tau K = \emptyset$ (1 point).

We next show that $H = K \cup \tau K$. Since $K \subseteq H$, $\tau \in H$, and H is closed under the operation, we conclude that $\tau K \subseteq H$. Therefore $K \cup \tau K \subseteq H$ (1 point). Conversely, suppose that $\sigma \in H$. Then σ is even or σ is odd.

Case 1: σ is even. Then $\sigma \in A_n$ which means that $\sigma \in A_n \cap H = K$.

Case 2: σ is odd. Since τ is odd, τ^{-1} is odd and therefore $\tau^{-1}\sigma$ is even. Since $\tau, \sigma \in H$, and H is a subgroup of G , $\tau^{-1}\sigma \in H$. Thus $\tau^{-1}\sigma \in K$ by Case 1. This means

$$\sigma = \text{Id}\sigma = (\tau\tau^{-1})\sigma = \tau(\tau^{-1}\sigma) \in \tau K.$$

We have shown that for all $\sigma \in H$ either $\sigma \in K$ or $\sigma \in \tau K$. Therefore $H \subseteq K \cup \tau K$ (**1 point**). Combining the two inclusions $H = K \cup \tau K$.

c) First of all $|H| = |K| + |\tau K|$ by part b) (**1 point**). Since $F : K \rightarrow \tau K$ is a set bijection, $|K| = |\tau K|$ by part a) (**1 point**). Combining these two statements we have $|H| = |K| + |\tau K| = |K| + |K| = 2|K|$ (**1 point**).

d) Suppose that H is subgroup of odd order. If H contains an odd permutation then $|H|$ is even by part c). Since $|H|$ is odd H can not contain an odd permutation. (**3 points**).