Homework #8 (week of 10/11–10/15)

Due *Monday*, 10/18/04 in class

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1. You may assume that the composition of one-one (respectively onto) functions is one-one (respectively onto). Thus the composition of one-one onto functions is one-one and onto. You may also assume that the inverse of a one-one onto function is again one-one and onto.

   a) Let $f : G \longrightarrow G'$ and $f' : G' \longrightarrow G''$ be isomorphisms of groups. Show that the composition $f' \circ f : G \longrightarrow G''$ is an isomorphism of groups.

   b) Let $f : G \longrightarrow G'$ be an isomorphism of groups. Show that the inverse function $f^{-1} : G' \longrightarrow G$ is an isomorphism of groups.

Let G be a group. An *automorphism of* G is a function $f : G \longrightarrow G$ which is a group isomorphism. The set of automorphisms of G is denoted by $\mathrm{Aut}(G)$.

   c) Show that $\mathrm{Aut}(G)$ is a group under composition. (Thus $\mathrm{Aut}(G)$ is a subgroup of the group of permutations on G under composition.)

For $a \in G$ let $\phi_a : G \longrightarrow G$ be the function defined by $\phi_a(x) = axa^{-1}$ for all $x \in G$.

   d) Show that $\phi_e = \mathrm{Id}$ and $\phi_a \circ \phi_b = \phi_{ab}$ for all $a, b \in G$.

   e) Show that $\phi_a \in \mathrm{Aut}(G)$ for all $a \in G$.

   f) Show that $f \circ \phi_a = \phi_{f(a)} \circ f$ for all $a \in G$ and $f \in \mathrm{Aut}(G)$.

2. Let $G = S_3$, $H = <(1\ 2)>$, and $K = <(1\ 2\ 3)>$.

   a) Find all left cosets of H in G and list their elements.

   b) Find all right cosets of H in G and list their elements.

c) Repeat parts a) and b) for K.

d) Find all subgroups of G and list their elements. [Hint: Lagrange's Theorem and one of its consequences may prove very helpful.]

3. Let p be a prime integer and G be a group of order p^2.

a) Show that G is cyclic or that $a^p = e$ for all $a \in G$.

b) Let $p = 2$ and suppose that $G = \{e, a, b, c\}$ is not cyclic. Show that there is one possibility for the Cayley table for G, where the elements of G are listed as above.

c) Find a non-cyclic subgroup of S_4 of order 4 and construct its Cayley table.