Hour Test I 10/08/04 Solution

Radford 10/10/04

1. (20 points total) Let $\imath \in I$. Since $H_\imath$ is a subgroup of G it follows that $e \in H_\imath$. Therefore $e \in \cap_{\imath \in I} H_\imath = K$ which implies that $K \neq \emptyset$ (**5 points**). Now let $a, b \in K$. Then $a, b \in H_\imath$ for all $\imath \in I$. Since the $H_\imath$'s are subgroups of G, by the 1-Step Subgroup Test $a^{-1}b \in H_\imath$ for all $\imath \in I$. Therefore $a^{-1}b \in \cap_{\imath \in I} H_\imath = K$ which concludes the proof that K is a subgroup of G by the 1-Step Subgroup Test (**15 points**). If the 2-Step Subgroup Test is used then closure, inverses 15 points also.
Comments: Induction does not work since the family of subgroups could very well be infinite.

2. (30 points total)

a) Since $45 = 3^2 \cdot 5$ has six divisors, which incidently are $1, 3, 5, 9, 15, 45$, the group G has *six* subgroups (**4 points**).

b)

Subgroup	Order	A generator
$<a>$	45	a^1
$<a^3>$	15	a^3
$<a^5>$	9	a^5
$<a^9>$	5	a^9
$<a^{15}>$	3	a^{15}
$<a^{45}>$	1	$a^0 = e$

(**8 points**).

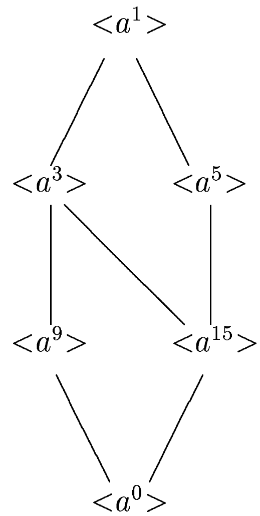
c) Since the integers $1 \leq k \leq 15$ which are relatively prime to 15 are the eight integers $1, 2, 4, 7, 8, 11, 13, 14$, the generators of the subgroup $<a^3>$ of G of order 15 are $(a^3)^k$ for these values of k. Thus the answer is:

$$a^3, a^6, a^{12}, a^{21}, a^{24}, a^{33}, a^{39}, a^{42} \quad (\textbf{6 points}).$$

d) Since $\gcd(45, 250) = \gcd(3^2{\cdot}5, 2{\cdot}5^3) = 5$ it follows that $d = 5$ (**3 points**). Thus $<a^{250}> = <a^5>$ and has the nine elements

$$e = a^0, a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40} \quad (\textbf{3 points}).$$

e)



(**6 points**).

3. (25 points total) $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^0 & 0 \\ 0 & 3^0 \end{pmatrix} \in H$, here $m = n = b = 0$, so $H \neq \emptyset$ (**5 points**). For $a, b, d \in \boldsymbol{R}$, where $a, d \neq 0$, we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -ba^{-1}d^{-1} \\ 0 & d^{-1} \end{pmatrix}.$$

(**8 points for an inverse calculation**). In particular $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$. Thus $H \subseteq G$. Since

$$\begin{pmatrix} 2^m & b \\ 0 & 3^n \end{pmatrix}^{-1} \begin{pmatrix} 2^{m'} & b' \\ 0 & 3^{n'} \end{pmatrix} = \begin{pmatrix} 2^{-m+m'} & b'' \\ 0 & 3^{-n+n'} \end{pmatrix}^{-1} \in H,$$

where $b'' = 2^{-m}b' - b2^{-m}3^{-n+n'}$, we conclude that H is a subgroup of G by the 1-step subgroup test (**12 points**). If the 2-Step Subgroup Test is used then closure, inverses 12 points also.

4. (25 points total) $f = (1\ 3\ 5\ 6)(2\ 4\ 6\ 9\ 7)(7\ 8\ 9)$ so a) $f = (1\ 3\ 5\ 4)(2\ 6\ 9)(7\ 8)$ (**10 points**). b) Using part a) we write $f = (5\ 4)(3\ 4)(1\ 4)(6\ 9)(2\ 9)(7\ 8)$ (**5 points**). f is even since it can be written as a product of an even number (6) of 2-cycles (**5 points**). d) Using part a) we have the disjoint cycle decomposition $f^2 = (1\ 5)(3\ 4)(2\ 9\ 6)$ (**5 points**).