Notes on Permutation Groups

11/01/04 Radford

Throughout a, b, c, d are distinct positive integers as are $a_1, a_2, \ldots, a_r$.

Let $r \geq 2$. Then the cycle $(a_1 \cdots a_r)$ can be written as a product of two-cycles in two natural ways:

$$\begin{aligned} (a_1 \ \cdots \ a_r) &= (\mathbf{a_1}\ a_r)(\mathbf{a_1}\ a_{r-1}) \cdots (\mathbf{a_1}\ a_2) \\ &= (a_{r-1}\ \mathbf{a_r}) \cdots (a_2\ \mathbf{a_r})(a_1\ \mathbf{a_r}). \end{aligned} \tag{1}$$

We use boldface for emphasis. For $n \geq 2$ all permutations of S_n can be written as a product of two-cycles.

Proposition 1 *Let $n \geq 2$. Then every permutation $\sigma \in S_n$ can be written as a product of two-cycles of S_n.*

PROOF: Let $\sigma = \sigma_1 \cdots \sigma_s$ be a decomposition of σ into a product of disjoint cycles. Write each $\sigma_\imath$ whose length is at least two as a product of two-cycles. If no $\sigma_\imath$ has length at least 2 then $\sigma = \mathrm{Id}$. Since $n \geq 2$, in this case we may write $\sigma = (1\ 2)(1\ 2)$ as a product of two-cycles in S_n. □

Consider $\sigma = (4\ 5)(1\ 2\ 3\ 4)(3\ 5\ 7) \in S_{11}$. Note that σ can be regarded as an element of S_n for $n \geq 7$. Simply replacing each of the cycles by a product of two-cycles by the first way described above gives

$$\sigma = \Big((4\ 5)\Big)\Big((1\ 4)(1\ 3)(1\ 2)\Big)\Big((3\ 7)(3\ 5)\Big) = (4\ 5)(1\ 4)(1\ 3)(1\ 2)(3\ 7)(3\ 5).$$

On the other hand, following the steps suggested by the preceding proposition yields the product

$$\begin{aligned} \sigma &= (4\ 5)(1\ 2\ 3\ 4)(3\ 5\ 7) \\ &= (1\ 2\ 3\ 4)(5\ 7) \\ &= (1\ 4)(1\ 3)(1\ 2)(5\ 7) \end{aligned}$$

Observe that although the number of two-cycles in both products representing σ are different, these numbers are both even. Generally a permutation can not be written as a product of an even number of permutations and as a product of an odd number of permutations. We shall prove this, starting with the special case of the identity permutation.

Lemma 1 *Let $n \geq 2$ and suppose that* $\mathrm{Id} = \sigma_1 \cdots \sigma_r$ *is the product of two-cycles. Then r is even.*

PROOF: Suppose that Id is a product of r two-cycles. We will show that Id can be rewritten as the product of $r-2$ two-cycles (0 if $r=2$ in the first place). If $r-2$ is even then r is even. Thus the proof will follow by induction on r.

First of all, if $\mathrm{Id} = \sigma_1 \cdots \sigma_r$ is the product of two-cycles then $r \geq 2$. Let $\sigma_r = (c\ \mathbf{d})$ and consider the following equations (again we use boldface for emphasis):

$$\begin{aligned}
(c\ d)(c\ \mathbf{d}) &= \mathrm{Id}; \\
(a\ d)(c\ \mathbf{d}) &= (c\ \mathbf{d})(a\ c); \\
(c\ a)(c\ \mathbf{d}) &= (a\ \mathbf{d})(a\ c); \\
(a\ b)(c\ \mathbf{d}) &= (c\ \mathbf{d})(a\ b).
\end{aligned}$$

By virtue of the equations we see that a cancelation occurs in the product $\sigma_1 \cdots \sigma_r$, or two adjacent two-cycles may be replaced by two others, which may cancel, or in the new product a symbol **d** occurs for the first time further to the left than before. Thus we may assume such a **d** occurs for the first time in σ_1 or σ_2. Since $\mathrm{Id}(d) = d$ necessarily d occurs in σ_2 and thus $\sigma_1 = \sigma_2$. This means $\mathrm{Id} = \sigma_3 \cdots \sigma_r$ is the product of $r-2$ two-cycles. □

Here are two examples which illustrate the mechanics of the proof.

$$\begin{aligned}
\mathrm{Id} &= (2\ 3)(1\ 4)(3\ 4)(1\ 4)(2\ 3)(1\ 2)(1\ 5)(1\ 5) \\
&= (2\ 3)(1\ 4)(3\ 4)(1\ 4)(2\ 3)(1\ 2)\boxed{(1\ 5)(1\ \mathbf{5})} \\
&= (2\ 3)(1\ 4)(3\ 4)(1\ 4)(2\ 3)(1\ 2)\boxed{} \\
&= (2\ 3)(1\ 4)(3\ 4)(1\ 4)(2\ 3)(1\ 2)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Id} &= (2\,3)(1\,4)(3\,4)(1\,4)(2\,3)(1\,2) \\
&= (2\,3)(1\,4)(3\,4)(1\,4)\boxed{(3\,2)(1\,\mathbf{2})} \\
&= (2\,3)(1\,4)(3\,4)(1\,4)\boxed{(1\,\mathbf{2})(1\,3)} \\
&= (2\,3)(1\,4)(3\,4)\boxed{(1\,4)(1\,\mathbf{2})}(1\,3) \\
&= (2\,3)(1\,4)(3\,4)\boxed{(4\,\mathbf{2})(4\,1)}(1\,3) \\
&= (2\,3)(1\,4)\boxed{(3\,4)(4\,\mathbf{2})}(4\,1)(1\,3) \\
&= (2\,3)(1\,4)\boxed{(4\,3)(4\,\mathbf{2})}(4\,1)(1\,3) \\
&= (2\,3)(1\,4)\boxed{(3\,\mathbf{2})(3\,4)}(4\,1)(1\,3) \\
&= (2\,3)\boxed{(1\,4)(3\,\mathbf{2})}(3\,4)(4\,1)(1\,3) \\
&= (2\,3)\boxed{(3\,\mathbf{2})(1\,4)}(3\,4)(4\,1)(1\,3) \\
&= \boxed{(2\,3)(3\,\mathbf{2})}(1\,4)(3\,4)(4\,1)(1\,3) \\
&= \boxed{(3\,2)(3\,\mathbf{2})}(1\,4)(3\,4)(4\,1)(1\,3) \\
&= \Box(1\,4)(3\,4)(4\,1)(1\,3) \\
&= (1\,4)(3\,4)(4\,1)(1\,3).
\end{aligned}
$$

Theorem 1 *Let $n \geq 1$ and suppose that $\sigma \in S_n$ is written as $\sigma = \sigma_1 \cdots \sigma_r = \sigma'_1 \cdots \sigma'_{r'}$, as a product of two-cycles in two different ways. Then both r and r' are even or both are odd.*

PROOF: Since two-cycles have order 2 they are their own inverses. Thus

$$
\begin{aligned}
\mathrm{Id} &= \sigma\sigma^{-1} = (\sigma_1 \cdots \sigma_r)((\sigma'_1 \cdots (\sigma'_{r'})^{-1} \\
&= (\sigma_1 \cdots \sigma_r)((\sigma'_{r'})^{-1} \cdots (\sigma'_1))^{-1}) \\
&= (\sigma_1 \cdots \sigma_r)(\sigma'_{r'} \cdots \sigma'_1) \\
&= \sigma_1 \cdots \sigma_r \sigma'_{r'} \cdots \sigma'_1.
\end{aligned}
$$

By Lemma 1 the sum $r + r'$ is even. $\Box$

Suppose that $\sigma \in S_n$. If $n = 1$ then σ is called even. Suppose that $n \geq 2$. Then σ can be written as the product of two-cycles. If σ can be written as a product of an even number of permutations then σ is a called even; otherwise σ is called odd.

The theorem has the following implications for an even permutation σ when $n \geq 2$. By definition σ is *some* product of an even number of two-cycles. Suppose that σ is written as a product of two-cycles in another way. Then the number of two-cycles in this product must be even also. Consequently if σ is an odd permutation then any realization of σ as a product of two-cycles must have an odd number of factors.

Let $n \geq 2$. Then every permutation is a product of two-cycles. Let $n \geq 3$ and suppose that $\sigma \in S_n$ is even. Then σ is the product of and even number of two-cycles, say $\sigma = \sigma_1 \cdots \sigma_r$. We may thus *pair* the two-cycles and write

$$\sigma = (\sigma_1 \sigma_2) \cdots (\sigma_{r-1} \sigma_r).$$

The calculations

$$(a\ b)(a\ b) = \mathrm{Id} = (a\ b\ c)(a\ c\ b),$$

$$(a\ b)(a\ c) = (a\ c\ b),$$

and

$$(a\ b)(c\ d) = (a\ c\ d)(a\ b\ d)$$

show that the product of a pair of two-cycles is three-cycle or product of three-cycles. Now three-cycles are even by (2). Thus

Corollary 1 *Let $n \geq 3$. Then $\sigma \in S_n$ is even if and only if σ is the product of three-cycles.* $\square$