MATH 425 **Written Homework 1 Solution** **Radford** **02/05/08**

1. (**20 points**)

(a) P is the set (subspace) of solutions to the system of linear equations $3x+2y-z+5w=0$ and consists of all vectors in $\mathbf{R}^4$ perpendicular to $\mathbf{u}$. (**10**)

(b) Row reduction yields $x+\frac{2}{3}y-\frac{1}{3}z+\frac{5}{3}w=0$ and thus

$$\begin{aligned} x &= -\frac{2}{3}y+\frac{1}{3}z-\frac{5}{3}w \\ y &= 1y+0z+0w \\ z &= 0y+1z+0w \\ w &= 0y+0z+1w \end{aligned}$$

which in vector form is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = y\begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -\frac{5}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus a basis for P is

$$\{\begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}\},$$

(**10**) or clearing fractions

$$\{\begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 0 \\ 3 \end{pmatrix}\}.$$

Comment: There are many possible answers.

2. (**20 points**) First some very basic observations about matrix multiplication. Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ be the standard basis for $\mathbf{R}^n$. Suppose that A is an $m\times n$ matrix (with real coefficients). Then

$$A\mathbf{e}_j \qquad \text{is the } j^{th} \text{ column of } A.$$

Suppose that A is an $n\times m$ matrix. Then

$$\mathbf{e}_i^t A \qquad \text{is the } i^{th} \text{ row of } A.$$

Now suppose that $m = n$, write $A = (a_{i\,j})$, and consider the bilinear form on $\mathbf{R}^n$ defined by

$$<\mathbf{u},\, \mathbf{v}> = \mathbf{u}^t A \mathbf{v} \tag{1}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Then the calculation

$$<\mathbf{e}_i,\, \mathbf{e}_j> = \mathbf{e}_i^t A \mathbf{e}_j = \mathbf{e}_i^t (A\mathbf{e}_j) = \mathbf{e}_i^t \begin{pmatrix} a_{1\,j} \\ \vdots \\ a_{n\,j} \end{pmatrix} = a_{i\,j},$$

where the latter is identified with the $1{\times}1$ matrix with entry $a_{i\,j}$, shows that

$$<\mathbf{e}_i,\, \mathbf{e}_j> = a_{i\,j}$$

for all $1 \leq i, j \leq n$.

Suppose that $<\ ,\ \ >$ is symmetric. Then $a_{i\,j} = <\mathbf{e}_i,\, \mathbf{e}_j> = <\mathbf{e}_j,\, \mathbf{e}_i> = a_{j\,i}$ for all $1 \leq i, j \leq n$ shows that A is symmetric.

Comment: Some solutions ended "$\mathbf{u}^t A \mathbf{v} = \mathbf{u}^t A^t \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, and therefore $A = A^t$." There is a significant gap in this proof.

3. (**20 points**) Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines an inner product on $\mathbf{R}^2$ by (1). Then A is symmetric by Exercise 4.2.7. Thus $c = b$. Using the solution to Exercise 4.2.7 we observe that

$$0 < <\mathbf{e}_1,\, \mathbf{e}_1> = a_{1\,1} = a \qquad \text{and} \qquad 0 < <\mathbf{e}_2,\, \mathbf{e}_2> = a_{2\,2} = d.$$

Since $b = c$ and $a, d \neq 0$ (as they are positive), it follows that

$$< \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > = ax^2 + 2bxy + dy^2 = a(x + \frac{b}{a}y)^2 + (d - \frac{b^2}{a})y^2 \tag{2}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$. By virtue of (2) we have

$$0 < < \begin{pmatrix} -b \\ a \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix} > = (d - \frac{b^2}{a})a^2$$

from which we deduce $d - \dfrac{b^2}{a} > 0$, or equivalently $ad - b^2 > 0$, since $a^2 > 0$. (**10**)

Conversely, suppose that $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ where $a, d, ad - b^2 > 0$. Since A is symmetric (1) defines a symmetric bilinear form on $\mathbf{R}^2$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$. Then (2) holds, and the right hand expression is a non-negative real number since it is the sum of products of non-negative real numbers. Suppose that the right hand expression is 0. Since the two summands are non-negative, it follows that

$$a(x + \frac{b}{a}y)^2 = (d - \frac{b^2}{a})y^2 = 0,$$

and thus

$$(x + \frac{b}{a}y)^2 = y^2 = 0$$

as $a, d - \frac{b^2}{a} \neq 0$. Therefore $y = 0$, and hence $x^2 = 0$. We have shown that $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$. (**10**)

4. (**20 points**) Suppose that A is a 2×2 matrix with real coefficients as in Exercise 4.2.8. Then A determines an inner product on $\mathbf{R}^2$ by (1). Such a matrix is $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where a, d are positive real numbers. Observe that

$$< \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} > = ax_1y_1 + dx_2y_2$$

for all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbf{R}^2$.

(a) $||\mathbf{u}|| = \sqrt{a+d}$. Thus take $a = 2000$, $d = 1$ and $a = 1$, $d = 2000$. In either case $||\mathbf{u}|| = \sqrt{2001}$. These choices give *different* inner products; indeed in the first case $|| \begin{pmatrix} 1 \\ 0 \end{pmatrix} || = \sqrt{2000}$ and in the second $|| \begin{pmatrix} 1 \\ 0 \end{pmatrix} || = 1$. (**6**)

(b) and (c). $<\mathbf{u}, \mathbf{v}> = a - d$ and $||\mathbf{u}|| = \sqrt{a+d} = ||\mathbf{v}||$. Thus

$$\cos\theta = \frac{<\mathbf{u}, \mathbf{v}>}{||\mathbf{u}||\,||\mathbf{v}||} = \frac{a-d}{a+d}.$$

For part (a) we need to solve $\frac{a-d}{a+d} = \frac{1}{2}$, or equivalently $a = 3d$. Take $d = 1$, $a = 3$ for example. (**7**) For part (b) we need to solve $\frac{a-d}{a+d} = \frac{\sqrt{3}}{2}$, or equivalently $a(2-\sqrt{3}) = d(2+\sqrt{3})$. Take $d = 1$, $a = \frac{2+\sqrt{3}}{2-\sqrt{3}}$ for example. (**7**)

Comment: Some students assumed that $||\mathbf{u}|| = \sqrt{2} = ||\mathbf{v}||$ in solving parts (b) and (c). This is the case for the standard inner product. These lengths depend on the choice of a and d.

5. (**20 points**) Let $\mathbf{u}, \mathbf{v} \in V$. From the calculation

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= <\mathbf{u}+\mathbf{v},\, \mathbf{u}+\mathbf{v}> \\ &= <\mathbf{u}+\mathbf{v},\, \mathbf{u}> + <\mathbf{u}+\mathbf{v},\, \mathbf{v}> \\ &= (<\mathbf{u},\, \mathbf{u}> + <\mathbf{v},\, \mathbf{u}>) + (<\mathbf{u},\, \mathbf{v}> + <\mathbf{v},\, \mathbf{v}>) \\ &= ||\mathbf{u}||^2 + 2<\mathbf{u}, \mathbf{v}> + ||\mathbf{v}||^2 \end{aligned}$$

we conclude that

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + 2<\mathbf{u}, \mathbf{v}> + ||\mathbf{v}||^2. \tag{3}$$

"If". Suppose that $<\mathbf{u},\, \mathbf{v}> = 0$. Then $2<\mathbf{u},\, \mathbf{v}> = 0$ and thus $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ by (3). (**10**)

"Only if". Suppose that $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$. Then $2<\mathbf{u},\, \mathbf{v}> = 0$ by (3) and hence $<\mathbf{u},\, \mathbf{v}> = 0$. (**10**)