MATH 425 **Written Homework 3 Solution** **Radford** **02/12/08**

1. (**20 points**) $A = \begin{pmatrix} 4 & 98 & -7 \\ 0 & 3 & 0 \\ 2 & 42 & -5 \end{pmatrix}$. Thus the characteristic polynomial of A is

$$\begin{aligned} c_A(x) &= \mathrm{Det}(A - x\mathrm{I}_3) \\ &= \begin{vmatrix} 4-x & 98 & -7 \\ 0 & 3-x & 0 \\ 2 & 42 & -5-x \end{vmatrix} \\ &= (3-x)\begin{vmatrix} 4-x & -7 \\ 2 & -5-x \end{vmatrix} \\ &= (3-x)(-(4-x)(5+x)+14) \\ &= (3-x)(x^2+x-6) \\ &= (3-x)(x+3)(x-2) \end{aligned}$$

which means that the eigenvalues for A are $\lambda = -3, 2, 3$. Thus A is diagonalizable by Corollary 5.2.1. (**5**)

To find S we find a basis of eigenvectors for each eigenvalue.

$\lambda = -3$: By row reduction $A - \lambda\mathrm{I}_3 = A + 3\mathrm{I}_3 = \begin{pmatrix} 7 & 98 & -7 \\ 0 & 6 & 0 \\ 2 & 42 & -2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 7 & 0 & -7 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{pmatrix} \longrightarrow$

$\cdots \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which means $N(A + 3\mathrm{I}_3)$ consists of the solutions to $\begin{matrix} x &=& z \\ y &=& 0 \\ (z &=& z) \end{matrix}$

which in vector form can be expressed as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for all $z \in \mathbf{R}$. Thus $\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis of the subspace of eigenvectors of A for $\lambda = -3$. (**2**)

$\lambda = 2$: By row reduction $A - \lambda\mathrm{I}_3 = A - 2\mathrm{I}_3 = \begin{pmatrix} 2 & 98 & -7 \\ 0 & 1 & 0 \\ 2 & 42 & -7 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 2 & 0 & -7 \\ 0 & 1 & 0 \\ 2 & 0 & -7 \end{pmatrix} \longrightarrow$

$\cdots \longrightarrow \begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which means $N(A - 2\mathrm{I}_3)$ consists of the solutions to $\begin{matrix} x &=& (7/2)z \\ y &=& 0 \\ (z &=& z) \end{matrix}$

which in vector form can be expressed as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 7/2 \\ 0 \\ 1 \end{pmatrix}$ for all $z \in \mathbf{R}$. Thus

$\{\begin{pmatrix} 7/2 \\ 0 \\ 1 \end{pmatrix}\}$, and also $\{\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}\}$, is a basis of the subspace of eigenvectors of A for $\lambda = 2$. (**2**)

$\lambda = 3$: By row reduction $A - \lambda I_3 = A - 3I_3 = \begin{pmatrix} 1 & 98 & -7 \\ 0 & 0 & 0 \\ 2 & 42 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 98 & -7 \\ 0 & 0 & 0 \\ 1 & 21 & -4 \end{pmatrix} \longrightarrow$ $\begin{pmatrix} 0 & 77 & -3 \\ 0 & 0 & 0 \\ 1 & 21 & -4 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 0 & -245/77 \\ 0 & 1 & -3/77 \\ 0 & 0 & 0 \end{pmatrix}$ which means $N(A - 3I_3)$ consists of the solutions to $\begin{matrix} x & = & (245/77)z \\ y & = & (3/77)z \\ (z & = & z) \end{matrix}$ which in vector form can be expressed as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 245/77 \\ 3/77 \\ 1 \end{pmatrix}$ for all $z \in \mathbf{R}$. Thus $\{\begin{pmatrix} 245/77 \\ 3/77 \\ 1 \end{pmatrix}\}$, and also $\{\begin{pmatrix} 245 \\ 3 \\ 77 \end{pmatrix}\}$, is a basis of the subspace of eigenvectors of A for $\lambda = 3$. (**3**)

Let $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. (**4**) Natural choices for S are $S = \begin{pmatrix} 1 & 7/2 & 245/77 \\ 0 & 0 & 3/77 \\ 1 & 1 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 7 & 245 \\ 0 & 0 & 3 \\ 1 & 2 & 77 \end{pmatrix}$. (**4**)

2. (**20 points**) $c_A(x) = \begin{vmatrix} 3-x & 7 & 5 & 2 \\ 0 & 2-x & 9 & 8 \\ 0 & 0 & 3-x & 1 \\ 0 & 0 & 0 & 2-x \end{vmatrix} = (3-x)^2(2-x)^2$. Thus $\lambda = 2, 3$ are the eigenvalues of A. (**4**) To compute the eigenvectors belonging to $\lambda = 2$ we use row reduction $A - \lambda I_4 = A - 2I_4 = \begin{pmatrix} 1 & 7 & 5 & 2 \\ 0 & 0 & 9 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 7 & 0 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$ $\begin{pmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; thus the eigenvectors for $\lambda = 2$ are given by $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = y\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, the subspace of which has basis $\{\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$. (**6**)

(a) Since the characteristic polynomial of A is $c_A(x) = (x-2)^2(x-3)^2$ the algebraic multiplicity of $\lambda = 2$ is 2. (**3**)

(b) Since the subspace of eigenvectors of A for $\lambda = 2$ has dimension 1, the geometric multiplicity of $\lambda = 2$ is 1. (**3**)

(c) Since the algebraic and geometric multiplicities of one of the eigenvalues of A differ, A is not diagonalizable by Theorem 5.2.1. (**4**)

3. (**20 points**) A in an invertible $n \times n$ matrix with real entries.

(a) Suppose that $\lambda = 0$ is an eigenvalue for A. Then $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ for some non-zero $\mathbf{x} \in \mathbf{R}^n$; in particular $\mathbf{x} \in N(A)$. Since A is invertible $N(A) = (0)$, a contradiction. Therefore $\lambda \neq 0$. (**5**)

(b) By part (a) the eigenvalues for A are not zero. Thus since A^{-1} is also invertible (with inverse A) the eigenvalues for A^{-1} are not zero.

Let $0 \neq \lambda \in \mathbf{R}$ and $\mathbf{x} \in \mathbf{R}^n$. Then $A\mathbf{x} = \lambda\mathbf{x}$ if and only if $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$ if and only if $\lambda^{-1}\mathbf{x} = \lambda^{-1}(\lambda A^{-1}\mathbf{x}) = A^{-1}\mathbf{x}$. We have shown that

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{if and only if} \qquad A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}. \tag{1}$$

Thus if λ is an eigenvalue for A then λ is invertible and λ^{-1} is an eigenvalue for A^{-1}. (**4**)

Conversely, suppose that ρ is an eigenvalue for A^{-1}. Then $\lambda = \rho^{-1}$ is an eigenvalue for $(A^{-1})^{-1} = A$. Since $\rho = (\rho^{-1})^{-1} = \lambda^{-1}$, every eigenvalue for A^{-1} has the form λ^{-1} for some eigenvalue λ for A. (**4**)

(c) $\mathbf{x} \in N(A - \lambda \mathrm{I}_n)$ if and only if $A\mathbf{x} = \lambda\mathbf{x}$, and likewise $\mathbf{x} \in N(A^{-1} - \lambda^{-1}\mathrm{I}_n)$ if and only if $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$. Thus part (c) follows by (1). (**7**)

4. (**20 points**) $A = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$

(a) From the computation of the characteristic polynomial

$$\begin{aligned} c_A(x) &= \text{Det}\, A - \text{Trace}\, Ax + x^2 \\ &= ((1-a)(1-b) - ab) - (2-a-b)x + x^2 \\ &= (1-a-b) - (2-a-b)x + x^2 \\ &= (x-1)(x-(1-a-b)) \end{aligned}$$

we see $\lambda = 1$ is an eigenvalue for A (**5**) (as is $\lambda = 1 - a - b$).

(b) For $\lambda = 1$, by row reduction $A - \lambda\mathrm{I}_2 = A - \mathrm{I}_2 = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & -b/a \\ 0 & 0 \end{pmatrix}$ we see that the eigenvectors are $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} b/a \\ 1 \end{pmatrix}$, where $y \in \mathbf{R}$. Now $\mathbf{v}$ is a probability vector if and only if its entries are non-negative and add to 1. This means

$y(b/a+1) = 1$, or equivalently $y = \dfrac{a}{a+b}$, and therefore $\mathbf{v} = \begin{pmatrix} \dfrac{b}{a+b} \\ \\ \dfrac{a}{a+b} \end{pmatrix}$. This vector is indeed a probability vector. (**5**)

(c) By assumption $0 < a, b < 1$. The two eigenvalues $\lambda = 1$ and $\lambda = 1 - a - b$ are distinct since otherwise $a + b = 0$, a contradiction. Since $0 < a + b < 2$ we have that

$$-1 < 1 - (a+b) < 1. \qquad (2)$$

It is is not difficult to see $\{\begin{pmatrix} b \\ a \end{pmatrix}\}$ is a basis for the eigenvectors belonging to $\lambda = 1$ and that $\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a basis for the eigenvectors belonging to $\lambda = 1-a-b$. Set $D = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$, where $c = 1-a-b$, and $S = \begin{pmatrix} b & -1 \\ a & 1 \end{pmatrix}$. Then $A = SDS^{-1}$. Since $S^{-1} = \dfrac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -a & b \end{pmatrix}$, it follows that

$$A^n = SD^nS^{-1} = \frac{1}{a+b}\begin{pmatrix} b & -1 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -a & b \end{pmatrix} = \frac{1}{a+b}\begin{pmatrix} b+ac^n & b-bc^n \\ a-ac^n & a+bc^n \end{pmatrix};$$

hence

$$A^n = \frac{1}{a+b}\begin{pmatrix} b+ac^n & b-bc^n \\ a-ac^n & a+bc^n \end{pmatrix} \qquad (3)$$

for all $n \geq 0$. Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^n$. Then

$$A\mathbf{v} = \frac{1}{a+b}\begin{pmatrix} b+ac^n & b-bc^n \\ a-ac^n & a+bc^n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a+b}\begin{pmatrix} (b+ac^n)x + (b-bc^n)y \\ (a-ac^n)x + (a+bc^n)y \end{pmatrix}.$$

Now $\lim_{n\longrightarrow\infty} c^n = 0$ since $|c| < 1$, which follows by (2); thus $\lim_{n\longrightarrow\infty} A^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{b(x+y)}{a+b} \\ \\ \dfrac{a(x+y)}{a+b} \end{pmatrix}$. When $x + y = 1$ note that $\lim_{n\longrightarrow\infty} A^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{b}{a+b} \\ \\ \dfrac{a}{a+b} \end{pmatrix}$. (**5**)

(d) follows by (3) since $\lim_{n\longrightarrow\infty} c^n = 0$. (**5**)

5. (**20 points**) $A = \begin{pmatrix} 1/5 & 3/11 & 0 \\ 2/5 & 6/11 & 0 \\ 2/5 & 2/11 & 1 \end{pmatrix}$. Therefore the characteristic polynomial is

$$c_A(x) \;=\; \begin{vmatrix} 1/5-x & 3/11 & 0 \\ 2/5 & 6/11-x & 0 \\ 2/5 & 2/11 & 1-x \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} 1/5 - x & 3/11 \\ 2/5 & 6/11 - x \end{vmatrix} (1 - x) \\
&= \Big((1/5 - x)(6/11 - x) - (3/11)(2/5) \Big) (1 - x) \\
&= (-41/55x + x^2)(1 - x) \\
&= x(x - 41/55)(1 - x).
\end{aligned}
$$

(a) Thus the eigenvalues for A are $\lambda = 0, 41/55, 1$ which means that A is diagonalizable by Corollary 5.2.1. (**7**)

(b) We show that A has a unique stable probability vector. This is a result of row reduction:

$$
A - \mathrm{I}_3 \;=\; \begin{pmatrix} -4/5 & 3/11 & 0 \\ 2/5 & -5/11 & 0 \\ 2/5 & 2/11 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & -7/11 & 0 \\ 2/5 & -5/11 & 0 \\ 0 & 7/11 & 0 \end{pmatrix}
$$

$$
\longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 2/5 & -5/11 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

which shows that $\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis for the the eigenvectors for A belonging to $\lambda = 1$; that is solutions $\mathbf{v} \in \mathbf{R}^3$ to $A\mathbf{v} = \mathbf{v}$, the stable vectors for A. A stable vector, which has the form $a \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for some $a \in \mathbf{R}$, is a probability vector if and only if $a = 1$. Therefore A has a unique stable probability vector which is $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. (**8**) That $\mathbf{v}$ is a limiting distribution follows by part (b) of Theorem 5.5.1 and part (a) of Corollary 5.5.1. (**5**)