Math 516 **Fall 2006** **Radford**

Written Homework # 1

Due at the beginning of class 09/15/06 (Revised slightly 09/06/06)

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1. Let G be a group.

   a) Suppose that H, K are subgroups of G. Show that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

   b) Let I be a non-empty set and suppose that $\{H_i\}_{i \in I}$ is an indexed family of subgroups of G which satisfies the following condition: For all $i, j \in I$ there is an $\ell \in I$ such that $H_i, H_j \subseteq H_\ell$. Show that the union $H = \cup_{i \in I} H_i$ is a subgroup of G.

2. Suppose that G and G' are groups.

   a) Show that the Cartesian product of sets $G \times G'$ is a group where

   $$(g, g')(h, h') = (gh, g'h')$$

   for all $(g, g'), (h, h') \in G \times G'$.

   b) Suppose that $f : G \longrightarrow G'$ is a group isomorphism. Show that $|g| = |f(g)|$ for all $g \in G$.

   Now let $G = \mathbf{Z}_2 = \{0, 1\}$ and $V = G \times G$.

   c) Set $e = (0, 0)$, $a = (1, 0)$, $b = (0, 1)$, and $c = (1, 1)$. Write down the table for the group structure of V.

   d) Show that there is *no* isomorphism $f : V \longrightarrow \mathbf{Z}_4$.

3. Let G be a group.

a) Suppose that X is a non-empty set and $G \times X \longrightarrow X$ is a left action of G on X. Show that $x \sim y$ if and only if $y = g \cdot x$ for some $g \in G$ defines an equivalence relation on X and for $x \in X$ the equivalence class

$$[x] = G \cdot x,$$

the G-orbit of x.

Now suppose that H is a subgroup of G.

b) Show that $h \cdot g = hg$, the product of h and g in G, for all $h \in H$ and $g \in G$ defines a left action of the group H on the set G.

c) Show that $[x] = Hx$ for all $x \in G$, where $Hx = \{hx \mid h \in H\}$.

d) For $x, y \in G$ show that $f : [x] \longrightarrow [y]$ given by $f(hx) = hy$ for all $h \in H$ is a *well-defined* function which is a set bijection. [Note: $hy \in [y]$ for all $h \in H$. Well-defined means that if $h, h' \in H$ then $hx = h'x$ implies $hy = h'y$.]

e) Now suppose that G is *finite*. Show that $|H|$ divides $|G|$.

4. Let G, G', and G'' be groups.

a) Show that the identity map $1_G : G \longrightarrow G$ is an isomorphism.

b) Suppose that $f : G \longrightarrow G'$ and $f' : G' \longrightarrow G''$ are homomorphisms (respectively isomorphisms). Show that the composite $f' \circ f : G \longrightarrow G''$ is a homomorphism (respectively an isomorphism). [You may assume that the composition of bijective functions is bijective.]

c) Suppose that $f : G \longrightarrow G'$ is an isomorphism. Show that its composition inverse $f^{-1} : G' \longrightarrow G$ is an isomorphism.

d) Show that the set $\mathrm{Aut}(G)$ of all isomorphisms from G to itself (that is the set of all automorphisms of G) is a subgroup of $S_G = \mathrm{Sym}(G)$.

e) The symbolism $G \sim G'$ means there exists an isomorphism $f : G \longrightarrow G'$. Show that "$\sim$" satisfies the axioms of an equivalence relation;

   (a) $G \sim G$,

   (b) $G \sim G'$ implies $G' \sim G$,

   (c) $G \sim G'$ and $G' \sim G''$ implies $G \sim G''$.

5. Let G be a group and $\mathrm{Aut}\,(G)$ be the group of all automorphisms of G under function composition; see Exercise 4.d).

a) For $g \in G$ let $\sigma_g : G \longrightarrow G$ be the function defined by $\sigma_g(x) = gxg^{-1}$ for all $x \in G$. Show that $\sigma_g \in \mathrm{Aut}\,(G)$.

b) Let $\pi : G \longrightarrow \mathrm{Aut}\,(G)$ be the function defined by $\pi(g) = \sigma_g$ for all $g \in G$. Show that π is a group homomorphism. [Thus $g{\cdot}x = \pi(g)(x)$ defines a left action of G on itself.]