Math 517 **Spring 2007** **Radford**

Final Examination 05/11/07

Name (PRINT) ________________________________

(1) *Return* this exam copy. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *eight questions* on this exam. (5) Each question counts 25 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

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1. Let $\{G_i\}_{i\in I}$ be a family of groups. A *coproduct of the family* is a pair $(G, \{\jmath_i\}_{i\in I})$, where

(1) G is a group and $\jmath_i : G_i \longrightarrow G$ is a group homomorphism for all $i \in I$, and

(2) If $(G', \{\jmath'_i\}_{i\in I})$ is a pair which satisfies (1) then there is a unique group homomorphism $f : G \longrightarrow G'$ which satisfies $f \circ \jmath_i = \jmath'_i$ for all $i \in I$.

Suppose that $(G, \{\jmath_i\}_{i\in I})$ and $(G', \{\jmath'_i\}_{i\in I})$ are coproducts of the family of groups $\{G_i\}_{i\in I}$. Show that $G \simeq G'$ as groups.

2. Let S be a ring with unity and R, L be right, left ideals respectively of S. Show that there is a homomorphism of abelian groups $R \otimes_S L \longrightarrow S$ given by $r \otimes \ell \mapsto r\ell$.

3. Let R be a ring with unity and let A, B, P be left R-modules.

(1) If the composition of R-module homomorphisms $A \xrightarrow{\jmath} B \xrightarrow{\pi} A$ is Id_A, show that $B = \mathrm{Ker}\,\pi \oplus \mathrm{Im}\,\jmath$.

(2) Suppose that $0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} P \longrightarrow 0$ is a short exact sequence of left R-modules, where P is projective. Show that $B = \mathrm{Ker}\,\pi \oplus C$ for some submodule C of B.

4. Let R be a ring with unity all of whose left R-modules are projective. Show that all non-zero left R-modules are completely reducible.

5. Consider the real numbers $\sqrt[5]{17}$ and $\sqrt[7]{10}$. Find:

(1) $\mathrm{m}_{\mathbf{Q},\sqrt[5]{17}}(x)$;

(2) $\mathrm{m}_{\mathbf{Q}[\sqrt[7]{10}],\sqrt[5]{17}}(x)$.

Your answers must be fully justified.

******** over for problems 6–8 ********

6. Suppose that K is a finite Galois extension of F. Describe $\mathrm{G}(K/F)$ as an abstract group in the following situations:

(1) $[K : F] = 7$;

(2) $F = \mathbf{Q}$ and $K = \mathbf{Q}[\zeta]$, where $\zeta \in K$ is a primitive n^{th} of unity;

(3) $K = F[\sqrt[18]{a}]$, where $[K : F] = 18$ and F contains a primitive 18^{th} root of unity;

(4) $[K : F] = 6$ and K is a splitting field of an irreducible polynomial $f(x) \in F[x]$ of degree 3;

(5) K is a finite field of 11^{23} elements and F is the prime field of K.

You need not justify your answers.

7. Let $K = \mathbf{Q}[\imath, a, b]$ be the field extension of $\mathbf{Q}$ in $\mathbf{C}$ generated by $\imath \in \mathbf{C}$, a square root of -1, and real numbers $a = \sqrt[4]{28}$, $b = \sqrt[11]{57}$. You may assume that $[K : \mathbf{Q}] = 2{\cdot}4{\cdot}11 = 88$, $[\mathbf{Q}[a] : \mathbf{Q}] = 4$, $[\mathbf{Q}[b] : \mathbf{Q}] = 11$, and $\mathrm{Aut}(K) = \mathrm{Aut}(K/\mathbf{Q})$.

(1) Show that $\sigma(b) = b$ for all $\sigma \in \mathrm{Aut}(K)$. (Thus $\mathrm{Aut}(K) = \mathrm{Aut}(K/\mathbf{Q}) = \mathrm{Aut}(K/\mathbf{Q}[b])$.)

(2) Show that K is a Galois extension of $\mathbf{Q}[b]$ and find $|\mathrm{G}(K/\mathbf{Q}[b])|$. (In particular $\mathrm{Aut}(K) = \mathrm{Aut}(K/\mathbf{Q}[b]) = \mathrm{G}(K/\mathbf{Q}[b])$.)

(3) Explain why $\sigma(\imath) \in \{\imath, \imath^3\} = \Omega$ and $\sigma(a) \in \{a, \imath a, \imath^2 a, \imath^3 a\} = A$ for all $\sigma \in \mathrm{Aut}(K)$.

(4) Show that for all $\imath' \in \Omega$ and $a' \in A$ there is a $\sigma \in \mathrm{Aut}(K)$ such that $\sigma(\imath) = \imath'$ and $\sigma(a) = a'$.

(5) Find $\sigma, \tau \in \mathrm{Aut}(K)$ such that σ has order 4, τ has order 2, and $\tau\sigma\tau^{-1} = \sigma^3$. Of course, justify your calculations.

8. Let R be a ring with unity, let M be a left R-module and let N be a submodule of M. Use Zorn's Lemma to show that there exists a submodule N' of M maximal with respect to the property that $N \cap N' = (0)$.