Math 517 **Spring 2007** **Radford**

Written Homework # 2 Solution

02/23/07

Throughout R and S are rings with unity; $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the rings of rational, real, and complex numbers respectively.

1. (**20 points**) Regard $\mathbf{C}$ as a (left) $\mathbf{Z}$-module, $\mathbf{Q}$-module, and $\mathbf{R}$-module by multiplication in $\mathbf{C}$. Let $f : \mathbf{C} \longrightarrow \mathbf{C}$ be the function defined by $f(r + s\imath) = s + r\imath$ for all $r, s \in \mathbf{R}$. Prove or disprove:

(1) (**5**) f is a homomorphism of additive groups (that is a homomorphism of $\mathbf{Z}$-modules);

(2) (**5**) f is a homomorphism of $\mathbf{Q}$-modules;

(3) (**5**) f is a homomorphism of $\mathbf{R}$-modules;

(4) (**5**) f is a homomorphism of $\mathbf{C}$-modules.

To disprove a statement find one specific example for which the statement fails.

Solution: Let $z = r + s\imath$ and $z' = r' + s'\imath$ be complex numbers written in standard form and let $r'' \in \mathbf{R}$. The calculations

$$\begin{aligned} f(z + z') &= (f((r + r') + (s + s')\imath) \\ &= (s + s') + (r + r')\imath \\ &= (s + r\imath) + (s' + r'\imath) \\ &= f(z) + f(z') \end{aligned}$$

and

$$f(r''z) = f(r''r + r''s\imath) = r''s + r''r\imath = r''(s + r\imath) = r''f(z)$$

show that f is a homomorphism of left $\mathbf{R}$-modules, hence of left $\mathbf{Q}$-modules and $\mathbf{Z}$-modules since $\mathbf{Q}, \mathbf{Z} \subseteq \mathbf{R}$.

Since $f(\imath\imath) = f((-1) + 0\imath) = 0 + (-1)\imath = -\imath$ and $\imath f(\imath) = \imath f(0 + 1\imath) = \imath(1 + 0\imath) = \imath$, then function f is not a homomorphism of left $\mathbf{C}$-modules.

2. (**20 points**) Suppose that R is commutative, ${}_RL$, ${}_RM$, ${}_RN$, ${}_RL'$, and ${}_RM'$. Using results from ClassNotes:

(1) Show that there are R-module isomorphisms $R \otimes_R M \simeq M$ (**5**), $M \otimes_R R \simeq M$ (**5**), and $L \otimes_R (M \otimes_R N) \simeq (L \otimes_R M) \otimes_R N$ (**5**).

(2) (**5**) Suppose that $f : L \longrightarrow L'$ and $g : M \longrightarrow M'$ are left R-module homomorphisms. Show that $f \otimes g : L \otimes_R M \longrightarrow L' \otimes_R M'$ is a left R-module homomorphism.

Solution: Background for both parts. Let ${}_RL$, ${}_RM$, and ${}_RN$. Since R is commutative these are right R-modules as defined in ClassNotes. Also ${}_R(M \otimes_R N)$ by Lemma 2.1.6, where

$$r \cdot (m \otimes n) = r \cdot m \otimes n = m \cdot r \otimes n = m \otimes r \cdot n$$

for all $r \in R$, $\in M$, and $n \in N$.

Part (1). By Proposition 2.1.2 there is an isomorphism of abelian groups $f : M \longrightarrow M \otimes_R R$ given by $f(m) = m \otimes 1$ for all $m \in M$. Let $r \in R$ and $m \in M$. The calculation

$$f(r \cdot m) = r \cdot m \otimes 1 = r \cdot (m \otimes 1) = r \cdot f(m)$$

shows that f is a homomorphism of R-modules; hence an isomorphism of R-modules.

You may assume that the function $g : M \longrightarrow R \otimes_R M$ defined by $g(m) = 1 \otimes m$ for all $m \in M$ is an isomorphism of abelian groups. For $r \in R$ and $m \in M$ the calculation

$$g(r \cdot m) = 1 \otimes r \cdot m = 1r \otimes m = r1 \otimes m = r \cdot (1 \otimes m) = r \cdot g(m)$$

shows that g is a homomorphism of R-modules; hence is an isomorphism of R-modules.

There is an isomorphism of abelian groups

$$f : L\otimes_R(M\otimes_R N) \longrightarrow (L\otimes_R M)\otimes_R N$$

given by $f(\ell\otimes(m\otimes n)) = (\ell\otimes m)\otimes n$ by Proposition 2.1.9. The calculation

$$\begin{aligned} f(r\cdot(\ell\otimes(m\otimes n))) &= f((r\cdot\ell\otimes(m\otimes n))) \\ &= (r\cdot\ell\otimes m)\otimes n \\ &= r\cdot(\ell\otimes m)\otimes n \\ &= r\cdot((\ell\otimes m)\otimes n) \\ &= r\cdot f(\ell\otimes(m\otimes n)) \end{aligned}$$

for all $r \in R$, $\ell \in L$, $m \in M$, and $n \in N$ shows that f is a homomorphism of left R-modules; hence is an isomorphism.

Part (2). By Proposition 2.1.5 $f\otimes g$ is a homomorphism of abelian groups. Let $r \in R$, $m \in M$, and $n \in N$. The calculation

$$\begin{aligned} f\otimes g(r\cdot(m\otimes n)) &= f\otimes g(r\cdot m\otimes n) \\ &= f(r\cdot m)\otimes g(n) \\ &= r\cdot f(m)\otimes g(n) \\ &= r\cdot(f(m)\otimes g(n)) \\ &= r\cdot(f\otimes g(m\otimes n)) \end{aligned}$$

shows that $f\otimes g$ is a homomorphism of left R-modules.

3. (**20 points**) Suppose L_R, ${}_RM_S$, and ${}_SN$. Use results from ClassNotes, except Propostion 2.1.9, to establish that there is a homomorphism of abelian groups $F : L\otimes_R(M\otimes_S N) \longrightarrow (L\otimes_R M)\otimes_S N$ given by $F(\ell\otimes(m\otimes n)) = (\ell\otimes m)\otimes n$ as follows:

(1) (**5**) Let $\ell \in L$ be fixed. Show that there is a homomorphism of abelian groups $f_\ell : M\otimes_S N \longrightarrow (L\otimes_R M)\otimes_S N$ given by $f_\ell(m\otimes n) = (\ell\otimes m)\otimes n$ for all $m \in M$ and $n \in N$.

Solution: By definition of the tensor product, we need only show the function $\varphi_\ell : M\times N \longrightarrow (L\otimes_R M)\otimes_S N$ given by $\varphi_\ell(m, n) = (\ell\otimes m)\otimes n$

for all $m \in M$, $n \in N$, and $\ell \in L$, is S-balanced. This is established by the calculations

$$\begin{aligned}
\varphi_\ell(m+m',n) &= (\ell\otimes(m+m'))\otimes n \\
&= (\ell\otimes m+\ell\otimes m')\otimes n \\
&= (\ell\otimes m)\otimes n+(\ell\otimes m')\otimes n \\
&= \varphi(m,n)+\varphi_\ell(m',n),
\end{aligned}$$

$$\begin{aligned}
\varphi_\ell(m,n+n') &= (\ell\otimes m)\otimes(n+n') \\
&= (\ell\otimes m)\otimes n+(\ell\otimes m)\otimes n' \\
&= \varphi(m,n)+\varphi_\ell(m,n'),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_\ell(m\cdot s,n) &= (\ell\otimes m\cdot s)\otimes n \\
&= (\ell\otimes m)\cdot s\otimes n \\
&= (\ell\otimes m)\cdot\otimes s\cdot n \\
&= \varphi(m,s\cdot n)
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$.

(2) (**5**) Show that $f_\ell + f_{\ell'} = f_{\ell+\ell'}$ for all $\ell, \ell' \in L$.

Solution: Let $\ell, \ell' \in L$, $m \in M$, and $n \in N$. Then

$$\begin{aligned}
&(f_\ell+f_{\ell'})(m\otimes n) \\
&\quad= f_\ell(m\otimes n)+f_{\ell'}(m\otimes n) \\
&\quad= (\ell\otimes m)\otimes n+(\ell'\otimes m)\otimes n \\
&\quad= (\ell\otimes m+\ell'\otimes m)\otimes n \\
&\quad= ((\ell+\ell')\otimes m)\otimes n \\
&\quad= f_{\ell+\ell'}(m\otimes n)
\end{aligned}$$

shows that the homomorphisms of abelian groups $f_\ell + f_{\ell'}$ and $f_{\ell+\ell'}$ agree on group generators. Thus $f_\ell + f_{\ell'} = f_{\ell+\ell'}$.

(3) (**5**) Show that $f : L\times(M\otimes_S N) \longrightarrow (L\otimes_R M)\otimes_S N$ defined by $f(\ell, x) = f_\ell(x)$ for all $\ell \in L$ and $x \in M\otimes N$ is R-balanced.

Solution: For $\ell, \ell' \in L$ and $x \in M\otimes N$ we have

$$\begin{aligned} f(\ell+\ell', x) &= f_{\ell+\ell'}(x) \\ &= (f_\ell + f_{\ell'})(x) \\ &= (f_\ell(x) + f_{\ell'})(x) \\ &= f(\ell, x) + f(\ell', x) \end{aligned}$$

by part (2),

$$\begin{aligned} f(\ell, x+x') &= f_\ell(x+x') \\ &= f_\ell(x) + f_\ell(x') \\ &= f(\ell, x) + f(\ell, x') \end{aligned}$$

since f_ℓ is a group homomorphism. Writing $x = \sum_{i=1}^{v} m_i\otimes n_i$ and noting that $f_{\ell\cdot r}$ is a group homomorphism, we calculate

$$\begin{aligned} f(\ell\cdot r, x) &= f_{\ell\cdot r}(x) \\ &= \sum_{i=1}^{v} f_{\ell\cdot r}(m_i\otimes n_i) \\ &= \sum_{i=1}^{v}(\ell\cdot r\otimes m_i)\otimes n_i \\ &= \sum_{i=1}^{v}(\ell\otimes r\cdot m_i)\otimes n_i \\ &= \sum_{i=1}^{v} f(\ell, r\cdot m_i\otimes n_i) \\ &= f(\ell, r\cdot x). \end{aligned}$$

This completes our proof.

(4) (**5**) Using (3) deduce the existence of F.

Solution: In light of (3), by definition of the tensor product there is a homomorphism of abelian groups F as described.

4. (**20 points**) Prove Proposition 3.2.1, parts (2) and (3), and Theorem 3.2.2 from ClassNotes.

Solution: Proof of part (2) of Proposition 3.2.1. (**5**) Consider the restriction $\pi|_{B''} : B'' \longrightarrow A$. Since π is an R-module homomorphism π'' is also. Let $b \in B$. Then $b = b' {\oplus} b''$ for unique $b' \in B' = \mathrm{Ker}, \pi$ and $b'' \in B''$. The calculation

$$\pi(b) = \pi(b' + b'') = \pi(b') + \pi(b'') = 0 + \pi(b'') = \pi|_{B''}(b'')$$

shows that

$$\pi(b) = \pi|_{B''}(b'').$$

Since π is surjective and it follows that $\pi|_{B''}$ is surjective. Suppose that $b'' \in \ker \pi|_{B''}$. Then with $b = 0{\oplus}b''$ we see that $0 = \pi|_{B''}(b'') = \pi(b)$ which means $0{\oplus}b'' \in \mathrm{Ker}\, \pi = \mathrm{Ker}\, \pi{\oplus}(0)$. Therefore $b'' = 0$. We have shown that the restriction $\pi|_{B''} : B'' \longrightarrow A$ is an isomorphism of R-modules.

Consider $\jmath' = \pi|_{B''}^{-1} : A \longrightarrow B$. Then $\mathrm{Im}\, \jmath' = B''$; thus

$$(\pi{\circ}\jmath')(a) = \pi(\pi|_{B''}^{-1}(a)) = \pi|_{B''}(\pi|_{B''}^{-1}(a)) = a$$

for all $a \in A$ which means that $\pi{\circ}\jmath' = \mathrm{Id}_A$.

Proof of part (3) of Proposition 3.2.1. (**5**) First of all if $L = M{\oplus}N$ is the direct sum of submodules then the projection $p : L \longrightarrow N$ given by $p(m{\oplus}n) = n$ is a well-defined homomorphism of left R-modules. Well-defined follow from the fact that if $\ell \in L$ then $\ell = m{\oplus}n$ for unique $m \in M$ and $n \in N$. Now p is a module map since

$$p(m{\oplus}n + m'{\oplus}n') = p((m + m'){\oplus}(n + n')) = n + n' = p(m{\oplus}n) + p(m'{\oplus}n')$$

and

$$p(r{\cdot}(m{\oplus}n)) = p((r{\cdot}m){\oplus}(r{\cdot}n)) = r{\cdot}n = r{\cdot}p(m{\oplus}n)$$

for all $m{\oplus}n, m'{\oplus}n' \in L$ and $r \in R$.

Since $\jmath$ is injective it determines an isomorphism which we denote, by abuse of notation, $\jmath : A \longrightarrow \mathrm{Im}\, \jmath$. Write $B = B'{\oplus}\mathrm{Im}\, jmath = M{\oplus}N$. Let $\pi' : B \longrightarrow A$ be the composite $\pi' = \jmath^{-1}{\circ}p$. Then π' is a homomorphism of left R-modules and

$$\pi'{\circ}\jmath(a) = \jmath^{-1}(p(\jmath(a)) = \jmath^{-1}(p(0{\oplus}\jmath(a)) = \jmath^{-1}(\jmath(a)) = a$$

for all $a \in A$ shows that $\pi' \circ \jmath = \mathrm{Id}_A$. Now $\operatorname{Ker} \pi' = \operatorname{Ker} p = B'$ since $\jmath^{-1}$ is injective.

Proof of Theorem 3.2.2. (**10**) Parts (1) and (2) are reformulations of each other since $\operatorname{Ker} \pi = \operatorname{Im} \alpha$ by exactness and if $M = L \oplus N$ is the direct sum of submodules then $M = N \oplus L$ as well. Part (1) implies part (3) follows by part (2) of Proposition 3.2.1, and part (3) implies part (1) follows by part (1) of Proposition 3.2.1. Part (2) implies part (4) follows by part (3) of Proposition 3.2.1, and part (4) implies part (2) follows by part (1) of Proposition 3.2.1.

5. (**20 points**) Let D be an integral domain and let F be its field of quotients. We may assume that D is a subring of F and we regard F as a left D-module by multiplication in F.

(1) (**4**) Suppose that M and N are submodules of F. Show that $M \cap N = (0)$ implies $M = (0)$ or $N = (0)$.

Solution: Suppose $M \neq (0)$ and let $0 \neq m/n \in M$. Let $m'/n' \in N$. Then $nm'(m/n) = mm' = n'm(m'/n')$ means that $nm'(m/n) \in M \cap N = (0)$. Therefore $nm'(m/n) = 0$. Since F is an integral domain and $n, m/n \neq 0$ necessarily $m' = 0$. Thus $m'/n' = 0$ and therefore $N = (0)$.

We have shown that $M \neq (0)$ implies $N = (0)$. Therefore $M = (0)$ or $N = (0)$.

(2) (**4**) Suppose that F is a free D-module and let $\{m_i\}_{i \in I}$ be a basis for F. Show that $|I| = 1$.

Solution: Suppose that $m_1, m_2 \in F$ are two different elements which belong to a basis. Then $Dm_1 \cap Dm_2 = (0)$. For let $a \in Dm_1 \cap Dm_2$. Then $a = d_1 m_1$ for some $d_1 \in D$ and $a = d_2 m_2$ for some $d_2 \in D$. But then

$$0 = a - a = d_1 m_1 - d_2 m_2 = d_1 m_1 + (-d_2) m_2$$

which means $d_1 = -d_2 = 0$. Therefore $a = 0$.

Now $Dm_1, Dm_2 \neq (0)$ since $m_1 \in Dm_1$ and $m_2 \in Dm_2$. By part (1) we have a contradiction. Thus no basis for F has more than one element.

(3) (**4**) Show that F is a free D-module if and only if $D = F$.

Solution: D is a free left D-module with basis $\{1\}$. So F is a free D-module when $F = D$.

Suppose that F is a free left D-module. There is a basis for F, and it must have one element by part (3). Let $\{m\}$ be the basis and write $m = a/b = ab^{-1}$. We only need to show that $b^{-1} \in D$; for then $F = Dm \subseteq D \subseteq F$ means $F = D$.

Now $1/b^2 = cm = c(a/b)$ for some $c \in D$. Multiplying both sides of the equation by b^2 yields $1 = cab$. Therefore $b^{-1} = ca \in D$.

(4) (**4**) Suppose that P is a non-zero projective R-module and $f : F \longrightarrow P$ is a surjective R-module homomorphism. Show that f is an isomorphism. [Hint: Consider part (1) of Proposition 3.2.1 which you can use without proof.]

Solution: Since $F \xrightarrow{f} P \longrightarrow 0$ is exact, for $\mathrm{Id}_{\mathrm{P}} : P \longrightarrow P$ there is a homomorphism of left R-modules $\jmath : P \longrightarrow F$ such that $f \circ \jmath = \mathrm{Id}_P$. Therefore $F = \mathrm{Ker}\,\pi \oplus \mathrm{Im}\,\jmath$ by part (1) of ClassNotes Proposition 3.2.1. As $\jmath$ is injective (and f is surjective) by part (1) by the same, and $P \neq (0)$, by part (1) necessarily $\mathrm{Ker}\,\pi = (0)$. Therefore f is injective and thus is an isomorphism.

(5) (**4**) Show that there is no surjective homomorphism of D-modules $f : F \longrightarrow D$ unless $D = F$. [Hint: D is a free, hence projective, D-module.]

Solution: Suppose $f : F \longrightarrow D$ is an isomorphism of left D-modules. Since D is a free left D-module it is projective. Thus f is an isomorphism by part (4). Since D is a free left D-module F must be also. Thus $D = F$ by part (3).